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11

TERMINOLOGY

angle of inclination
composition
dilation
identity
inverse
linear transformation
matrix
reflection
rotation
slope
transformation
translation
vector

MATRICES

TRANSFORMATIONS IN THE PLANE

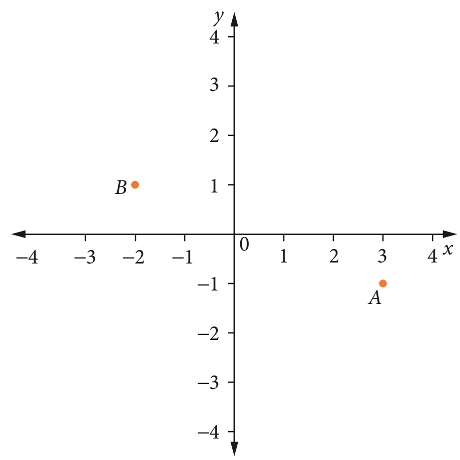
Prior learning

TRANSFORMATIONS IN THE PLANE

- translations and their representation as column vectors (ACMSM054)
- define and use basic linear transformations: dilations of the form $(x, y) \to (\lambda_1 x, \lambda_2 y)$, rotations about the origin and reflection in a line which passes through the origin, and the representations of these transformations by 2×2 matrices (ACMSM055)
- apply these transformations to points in the plane and geometric objects (ACMSM056)
- define and use composition of linear transformations and the corresponding matrix products (ACMSM057)
- define and use inverses of linear transformations and the relationship with the matrix inverse (ACMSM058)
- examine the relationship between the determinant and the effect of a linear transformation on area (ACMSM059)
- establish geometric results by matrix multiplications; for example, show that the combined effect of 2 reflections in lines through the origin is a rotation (ACMSM060) AC

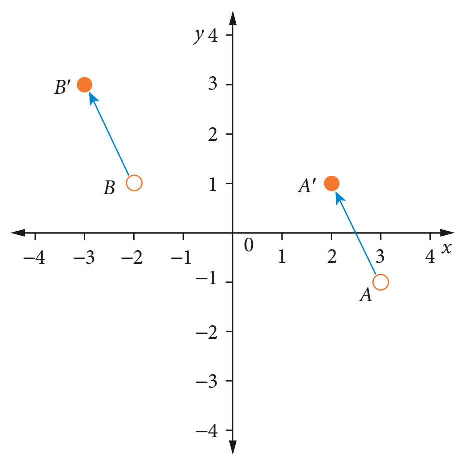
11.01 TRANSLATIONS

In Years 1–10 you examined transformations of geometric objects. You may also have looked at them in terms of changes in the coordinates of drawings in the Cartesian plane. The topic of transformations is very important in mathematics and its applications, such as the design and use of industrial robots.



In this chapter you will look at transformations as matrices. Points in the plane may be shown as column matrices, such as $A\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $B\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, when convenient. Points in the plane will also be written using standard coordinate notation, such as $A(3, -1)$ and $B(-2, 1)$.

In terms of movements, a **translation** is a *slide* of all the points in the plane the same distance in the same direction. The diagram on the right now shows a translation that moves the points in the plane *up* 2 and *left* 1. This is a translation of -2 in the x direction and 3 in the y direction.



The points $A\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $B\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are translated to the new points $A'\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $B'\begin{bmatrix} -3 \\ 3 \end{bmatrix}$.

You can write this as $A\begin{bmatrix} 3 \\ -1 \end{bmatrix} \to A'\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $B\begin{bmatrix} -2 \\ 1 \end{bmatrix} \to B'\begin{bmatrix} -3 \\ 3 \end{bmatrix}$.

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Example 1

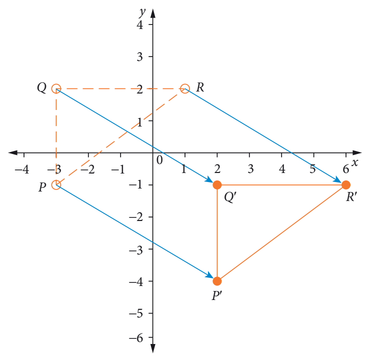
Show a translation of the triangle with vertices $P\begin{bmatrix}-3\\-1\end{bmatrix}$, $Q\begin{bmatrix}-3\\2\end{bmatrix}$ and $R\begin{bmatrix}1\\2\end{bmatrix}$ 3 down and 5 right as:

a a drawing in the plane

b a transformation of column matrices.

Solution

a The x-coordinates of each point increase by 5 and the y-coordinates decrease by 3. Draw the original points and the new points in the plane.



b 3 down and 5 right means that the x-coordinates increase by 5 and the y-coordinates decrease by 3.

$P\begin{bmatrix}-3\\-1\end{bmatrix} \to P'\begin{bmatrix}2\\-4\end{bmatrix}$, $Q\begin{bmatrix}-3\\2\end{bmatrix} \to Q'\begin{bmatrix}2\\-1\end{bmatrix}$

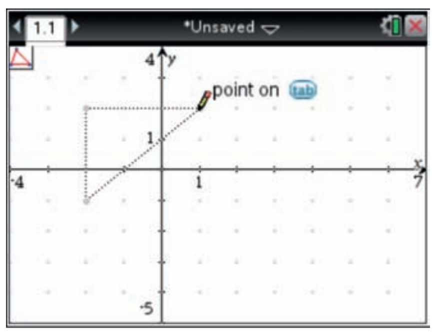
and $R\begin{bmatrix}1\\2\end{bmatrix} \to R'\begin{bmatrix}6\\-1\end{bmatrix}$

TI-Nspire CAS

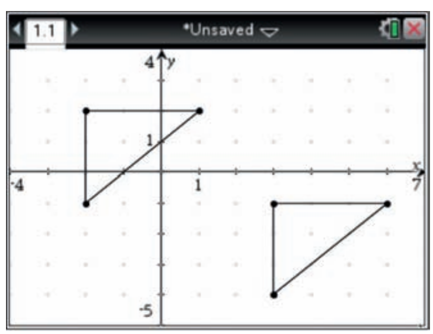
Use a graph page and Window settings of $-4 \le x \le 7$ and $-5 \le y \le 4$ with scales of 1 unit.

Use menu, 2: View, 6: Grid and 2: Dot Grid to show dots.

Use menu, 8: Geometry, 2: Shapes and Grid and 2: Triangle to draw the triangle by clicking the dots.



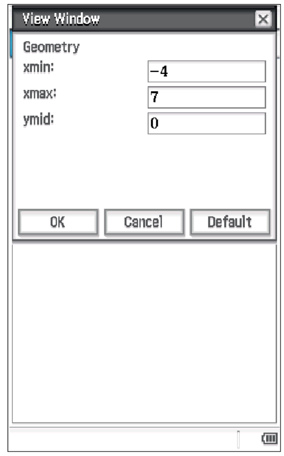
Now use menu, 8: Geometry, 5: Transformation and 3: Translation to do the translation by clicking the triangle, clicking a corner point and moving it 3 down and 5 right.



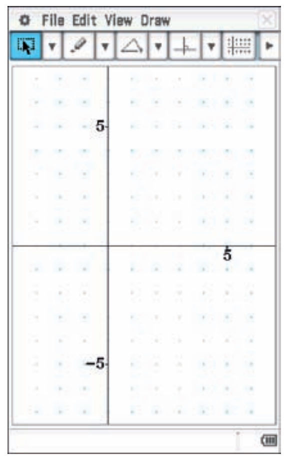
ClassPad

Use the Geometry menu.

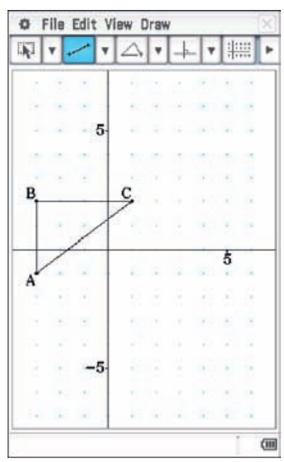
Under ⚙ tap **View Window** and set the fields in the window to make $-4 \leq x \leq 7$ and ymid = 0. ymid = 0 sets the middle of the screen to $y = 0$. When you have finished, tap **OK**.



Set the screen as shown by tapping ▦. You might have to tap it several times.



Tap **Draw** then **Basic Object** and **Line Segment** (or the ⟋ symbol) to draw the triangle by doing each side in order. (First tap the first point of the line segment then touch the point again and drag the stylus to the final point.) After these steps the triangle will appear as ΔABC.



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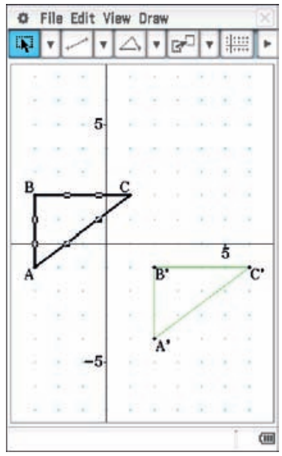
Tap select ( ) and select each side of the triangle. Use **Draw**, **Construct** and **Translation** to do the translation.
Enter $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ for 5 right and 3 down into the column matrix and tap OK.



The ClassPad automatically sets the y-scale to show the shapes accurately, but on the TI-Nspire CAS, if you want to show the shapes accurately, you need to set the x and y scales in a ratio of 2 : 1.

In Example **1**, you can transform the points by adding the column matrix $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ to each of the points.

Thus $P\begin{bmatrix} -3 \\ -1 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \end{bmatrix} = P'\begin{bmatrix} 2 \\ -4 \end{bmatrix}$, $Q\begin{bmatrix} -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \end{bmatrix} = Q'\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $R\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \end{bmatrix} = R'\begin{bmatrix} 6 \\ -1 \end{bmatrix}$.

The matrix $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ represents the translation 5 to the right and 3 down (5 in the x direction and –3 in the y direction).

IMPORTANT

A translation can be modelled as addition of column matrices. The column matrices representing points are translated to the new points by adding the column matrix representing the translation.

Example 2

Use column matrices to find the points that $A(2, 1)$, $B(4, 2)$, $C(5, 1)$, $D(3, -1)$ are transformed to by a translation of 3 left and 2 down.

Solution

Write the points as column matrices.

$A\begin{bmatrix} 2 \\ 1 \end{bmatrix}, B\begin{bmatrix} 4 \\ 2 \end{bmatrix}, C\begin{bmatrix} 5 \\ 1 \end{bmatrix}, D\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

Write the translation as a column matrix.

$\text{Translation} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

Add the translation to each column matrix.

$$A\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = A'\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$
$$B\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = B'\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$C\begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = C'\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$D\begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \end{bmatrix} = D'\begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

Write the answers.

The points become $A'(-1, -1)$, $B'(1, 0)$, $C'(2, -1)$, $D'(0, -3)$.

EXERCISE 11.01 Translations

Concepts and techniques

1 Example 1 Show each of the following translations as both a drawing in the plane and a transformation of column matrices.
   a The point $P(3, 4)$ is translated 3 down and 4 left.
   b The point $Q(-2, 1)$ is translated 4 right and 3 down.
   c The line segment $A(1, 3)$ $B(-1, -2)$ is translated 2 right and 1 down.
   d The line segment $F(-2, 1)$ $G(1, -2)$ is translated 3 right and 3 up.
   e The triangle $A(1, 2)$ $B(-2, -1)$ $C(2, 0)$ is translated 3 right and 2 down.

2 Example 2 Use matrices to find the points that each of the following are transformed to after a translation of 4 left and 3 up.
   a $A(-1, -5)$ b $B(2, 3)$ c $C(0, 0)$ d $D(1, -3)$ e $E(-3, 6)$

3 Use matrices to find the points to which each of the following are transformed to after a translation of 1 left and 2 down.
   a $A(3, 4)$ b $B(-5, -4)$ c $C(4, -3)$ d $D(6, -3)$ e $E(-4, 8)$

4 Use matrices to find the points that each of the following are transformed to after a translation of 5 right and 6 down.
   a $A(-4, 7)$ b $B(3, -3)$ c $C(-8, 2)$ d $D(-7, -5)$ e $E(-2, -4)$

5 The point $T(2, 4)$ is translated to $T'(-2, 3)$. Write the translation as a column matrix.

6 What translation changes the point $M(-3, 5)$ to $M'(4, 1)$?

7 What translation is necessary to reverse the translation $K(6, 3) \rightarrow K'(2, 9)$?

8 CAS Show a translation of the triangle $A(-2, -3)$ $B(1, 4)$ $C(3, 2)$ 3 left and 4 up.

9 CAS Show a translation of the pentagon $A(2, -1)$ $B(-2, 1)$ $C(0, 3)$ $D(2, 3)$ $D(3, 2)$ 3 down and 4 right.

Reasoning and communication

10 The translation $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is applied to the point $P(1, -2)$, followed by the translation $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.
   a What are the coordinates of the final point?
   b What is the combined translation?
   c Generalise this to state the overall effect of the two translations $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$.

11 Consider the combination of two translations, $\mathbf{S} * \mathbf{T}$, as an operation that produces another transformation.
   a Given a translation $\mathbf{T} = \begin{bmatrix} a \\ b \end{bmatrix}$, can you find an *identity* translation $\mathbf{I}$ so that $\mathbf{T} * \mathbf{I} = \mathbf{T}$?
   b Given a translation $\mathbf{T} = \begin{bmatrix} a \\ b \end{bmatrix}$, can you find an *inverse* translation $\mathbf{T}_i$ so that $\mathbf{T} * \mathbf{T}_i$ leaves every point in the same place by undoing the first translation?

12 Show that the order in which you combine two translations does not matter.

11.02 LINEAR TRANSFORMATIONS

There are many ways in which points and shapes can be transformed. For example, the transformation of points given by $P(x, y) \to P'(x^2, y^3)$ changes every point in the plane to a point that either remains on the y-axis or is on the right of the y-axis. It also changes shapes in different ways in the vertical and horizontal directions.

The transformation $P(x, y) \to P'(xy, x^2 + y^{-1})$ is of 'mind-boggling' complexity, twisting and folding shapes in weird ways.

A **linear transformation** is an important type of transformation. A linear transformation involves only linear functions of x and y and leaves straight lines as straight lines (or shrinks them to a single point), although it may change their position and direction. In computer graphics, linear transformations are used to map 3D objects to 2D images.

A transformation may change a shape into a different one. For example, a sideways stretch would change a square into a rectangle. However, some properties may not change.

IMPORTANT

A **linear transformation of the plane** changes the point $P(x, y)$ to $P'(ax + by, cx + dy)$, where a, b, c and d are constants (real numbers).

Example 3

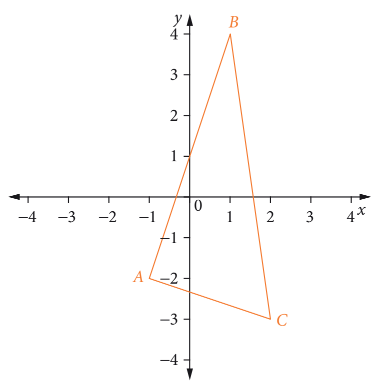
a Show that the triangle $A(-1, -2)$ $B(1, 4)$ $C(2, -3)$ is right-angled.
The triangle is transformed by the linear transformation $(x, y) \to (2x - y, 2x + y)$.

b Find the coordinates of the image $A'B'C'$.

c What is the shape of $A'B'C'$?

Solution

a Draw a diagram.
It looks as if $\angle CAB$ could be a right angle.



Find the vector for the side CA.
$\mathbf{CA} = (-1 - 2, -2 - (-3))$
$= (-3, 1)$

Find the vector for the side AB.
$\mathbf{AB} = (1 - (-1), 4 - (-2))$
$= (2, 6)$

Find $\mathbf{CA} \cdot \mathbf{AB}$.
$(-3, 1) \cdot (2, 6) = -3 \times 2 + 1 \times 6$
$= 0$

What does this mean?
Since $\mathbf{CA} \cdot \mathbf{AB} = 0$, $\mathbf{CA}$ and $\mathbf{AB}$ are perpendicular.

State the result.
ABC is a right-angled triangle.

b Apply $(x, y) \to (2x - y, 2x + y)$ to A.

$A(-1, -2) \to A'(2 \times (-1) - (-2),$
$2 \times (-1) + (-2))$

Simplify.

$A(-1, -2) \to A'(0, -4)$

Apply $(x, y) \to (2x - y, 2x + y)$ to B.

$B(1, 4) \to B'(2 \times 1 - 4, 2 \times 1 + 4)$

Simplify.

$B(1, 4) \to B'(-2, 6)$

Apply $(x, y) \to (2x - y, 2x + y)$ to C.

$C(2, -3) \to C'(2 \times 2 - (-3), 2 \times 2 + (-3))$

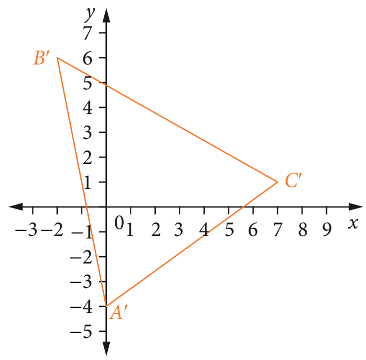
Simplify.

$C(2, -3) \to C'(7, 1)$

Write the result.

The points are $A'(0, -4)$, $B'(-2, 6)$, $C'(7, 1)$.

c Draw a diagram.
The shape *appears* to be a triangle that is *not* right-angled.
You need to check this.



Find the vector for the side $A'B'$.

$\mathbf{A'B'} = (-2, 6) - (0, -4)$
$= (-2, 10)$

Find the vector for the side $B'C'$.

$\mathbf{B'C'} = (7, 1) - (-2, 6)$
$= (9, -5)$

Find the vector for the side $C'A'$.

$\mathbf{C'A'} = (0, -4) - (7, 1)$
$= (-7, -5)$

Find their magnitudes.

$|\mathbf{A'B'}| = \sqrt{104}$,
$|\mathbf{B'C'}| = \sqrt{106}$,
$|\mathbf{C'A'}| = \sqrt{74}$

Check for a right-angle.

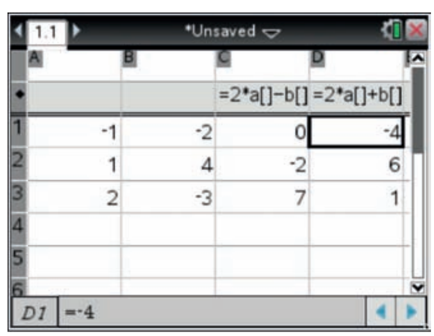
$|\mathbf{B'C'}|^2 = 106 \neq 104 + 74 = |\mathbf{A'B'}|^2 + |\mathbf{C'A'}|^2$

Write your conclusion.

The sides of $\triangle A'B'C'$ are all different and do not make a Pythagorean triple, so it is a scalene triangle.

TI-Nspire CAS

You can work out part **b** using your CAS calculator. Use the Lists & Spreadsheet page and enter the points into columns A and B. Then put =2a-b into column C, choosing column references.
Then put 2a+b into column D.



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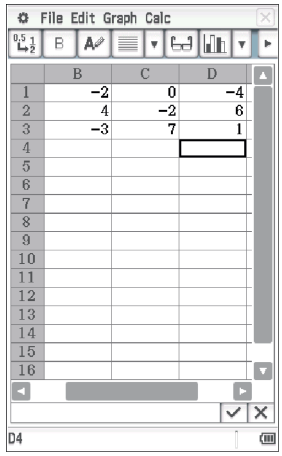
ClassPad

Use the **Spreadsheet** application.
Enter the points into columns A and B. Use one row per point. Enter the x coordinate of a point into column A and the y coordinate into column B.
Tap cell C1, press [=] 2 [×], tap cell A1, press [−] and tap cell B1 to enter the formula C1=2A1–B1. Press [EXE].
Tap C1, **Edit** and **Copy**, then tap C2 and slide down to C3. Tap **Edit** and **Paste** to copy the formula. Follow a similar process to enter the formula D1=2A1+B1 in the D column.



Part **a** of Example **3** could also have been done using gradients or Pythagoras' theorem. It shows that a linear transformation will not always change a geometric figure to an image that is exactly the same shape. However, the triangle *was* changed to another triangle. An obvious question is 'Does a linear transformation change a polygon to another polygon with the same number of sides?'

INVESTIGATION Linear transformations of polygons

In this investigation you will look at some transformations to determine whether polygons are always transformed to polygons with the same number of sides.

1 Consider the transformation $(x, y) \to (x + 2, \sqrt{y^2 - 4})$ and the hexagon $A(3, 2)$ $B(5, 5)$ $C(7, 2)$ $D(7, -2)$ $E(5, -3)$ $F(3, -2)$. Find the shape of the image $A'B'C'D'E'$ under the transformation. Does this show that linear transformations do not always produce an image of the same kind of polygon?

2 Consider the transformation $(x, y) \to (x, x)$. Find the shape of the rectangle $A(3, 2)$ $B(6, 6)$ $C(14, 0)$ $D(11, -4)$ under the transformation. Does this show that linear transformations do not always produce an image of the same kind of polygon?

3 What about the transformation $(x, y) \to (0, 0)$? It obviously shrinks everything to a single point, but is it linear?

4 Can you make up linear transformations that do not transform polygons to the same kind of polygon?

The process of working out the new points under a linear transformation is quite tedious. Consider the transformation in Example **3**, $(x, y) \to (2x - y, 2x + y)$. This can be written in the form of column vectors as $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 2x - y \\ 2x + y \end{bmatrix}$, treating the points (x, y) as position vectors. You can write this transformation as multiplication by a 2×2 matrix: $\begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ 2x + y \end{bmatrix}$.

Remember, matrix multiplication is *row* × *column*! The same method can be used to write *any* linear transformation $(x, y) \to (ax + by, cx + dy)$ as matrix multiplication.

IMPORTANT

The linear transformation that changes the point $P(x, y) \to P'(ax + by, cx + dy)$ is modelled by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. The matrix is multiplied by the column vector representing the point $P(x, y)$ to give the column vector representing $P'(x', y')$.

As they can be represented as matrices, we usually use a single capital letter to denote linear transformations, such as $\mathbf{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Example 4

a Express the linear transformation $\mathbf{T}$: $(x, y) \to (3x + 2y, 4x - y)$ as a matrix.

b Use the matrix to find the image of the isosceles triangle $A(2, 4)$ $B(3, 7)$ $C(4, 4)$ under the transformation $\mathbf{T}$.

Solution

a Write the rule. $(x, y) \to (ax + by, cx + dy)$ is modelled as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Substitute the values. $(x, y) \to (3x + 2y, 4x - y)$ is $\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$.

Write the answer.

$$\mathbf{T} = \begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}$$

b Apply to **a**.

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \end{bmatrix}$$

Apply to **b**.

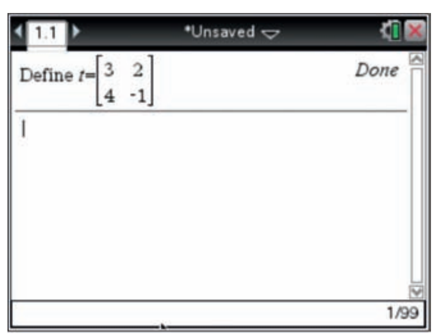
$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 23 \\ 5 \end{bmatrix}$$

Apply to **c**.

$$\begin{bmatrix} 3 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 20 \\ 12 \end{bmatrix}$$

TI-Nspire CAS

You can work this out on your CAS calculator by typing Define (or getting it from the [book icon] or [menu]) and then doing [menu] [enter] 7: Matrices and Vectors, 1: Create. All variables on the TI-Nspire are changed to lower case.



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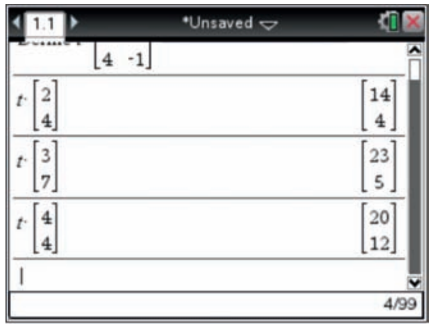
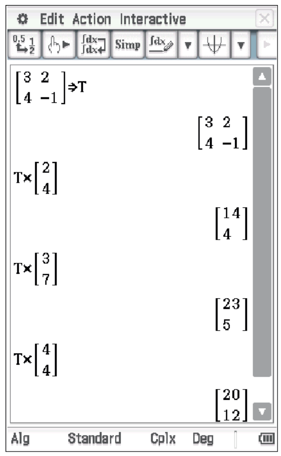
Multiply the matrix by each position vector in succession. Enter each position vector using menu 7: Matrices and Vectors, 1: Create, with 2 rows and 1 column.



ClassPad

Use the Main $\sqrt{\alpha}$ application.
The matrix and vector templates can be found by pressing Keyboard then tapping Math2.
Define the matrix **T** and then do the matrix multiplications.
You must include the multiplication symbol.



Write the answer.

$ABC \rightarrow A'(14, 4)\ B'(23, 5)\ C'(20, 12)$

What is the shape of $A'B'C'$ under the transformation **T** in Example **4**?

EXERCISE 11.02 Linear transformations

WS Linear transformations

Concepts and techniques

1 Example 3 **a** Show that the triangle $A(-1, -2)$ $B(-2, 1)$ $C(2, -3)$ is isosceles.
A transformation is given by $(x, y) \rightarrow (-2x - 2y, 3x + y)$.
b Find the coordinates of the image $A'B'C'$ under the transformation.
c What is the shape of $A'B'C'$?

2 **a** Show that $A(4, -3)$ $B(6, 2)$ $C(1, 4)$ $D(-1, -1)$ is a square.
A transformation is given by $(x, y) \rightarrow (2x - 3y, 3y - x)$.
b Find the coordinates of the image $A'B'C'D'$.
c What is the shape of $A'B'C'D'$?

3 **a** Show that $A(4, -3)$ $B(3, -1)$ $C(12, 6)$ $D(13, 4)$ is a parallelogram.
A transformation is given by $(x, y) \rightarrow (3x - y, 4y - 2x)$.
b Find the coordinates of the image $A'B'C'D'$.
c What is the shape of $A'B'C'D'$?

4 Example 4 **a** Express the linear transformation **T**: $(x, y) \rightarrow (2x - 4y, 4x + 3y)$ as a matrix.
b Use the matrix to find the image of the triangle $A(-1, 3)$ $B(2, 5)$ $C(3, -3)$ under the transformation **T**.

5 a Express the linear transformation $\mathbf{T}$: $(x, y) \to (x + y, -x - y)$ as a matrix.
b Use the matrix to find the image of the triangle $A(1, 2)$ $B(4, 4)$ $C(5, -4)$ under the transformation $\mathbf{T}$.

6 a CAS Express the linear transformation $\mathbf{T}$: $(x, y) \to (x - 2y, 2x - y)$ as a matrix.
b Use the matrix to find the image of the triangle $A(3, 5)$ $B(4, -2)$ $C(1, 1)$ under the transformation $\mathbf{T}$.

7 a CAS Express the linear transformation $\mathbf{T}$: $(x, y) \to (3y - 2x, 4x + 3y)$ as a matrix.
b Use the matrix to find the image of the parallelogram $A(-2, -3)$ $B(-1, -1)$ $C(-3, 2)$ $D(-4, 0)$ under the transformation $\mathbf{T}$.

8 a CAS Express the linear transformation $\mathbf{T}$: $(x, y) \to (5x - 2y, 3x + 4y)$ as a matrix.
b Use the matrix to find the image of $A(1, 5)$ $B(-2, 3)$ $C(5, -4)$ $D(7, 6)$ under the transformation $\mathbf{T}$.

Reasoning and communication

9 $A(2, 3)$ $B(4, 1)$ $C(5, -3)$ $D(3, -1)$ is a parallelogram.
a Find the images of the parallelogram under each of the transformations $\mathbf{T}_1 = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$ and $\mathbf{T}_2 = \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$.
b Find the shape of each image under each transformation.

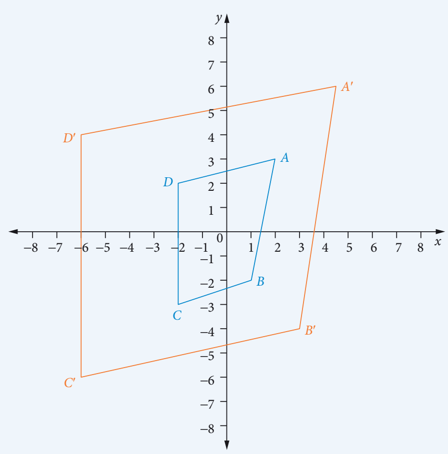
10 $A(2, 3)$ $B(3, 6)$ $C(5, 4)$ $D(4, 1)$ is a parallelogram. It is transformed by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
a Find the image under the transformation.
b Show that the image is also a parallelogram.

11.03 DILATIONS

A simple **dilation** stretches or compresses the plane in the x direction and/or the y direction. You have seen the effect of compression or stretching of functions on their graphs. Dilation does the same thing to geometric shapes.

INVESTIGATION Dilations

The dilation below magnifies the quadrilateral by a factor of 3 in the x direction and a factor of 2 in the y direction. The blue one is the original figure and the orange one is the magnified figure.



- What are the coordinates of A and A'?
- What are the coordinates of B and B'?
- How are the coordinates of the points for dilation by a factor of 3 related?
- Complete the transformation for dilation by a factor of 3 below.

$(x, y) \to$ (__________, __________)

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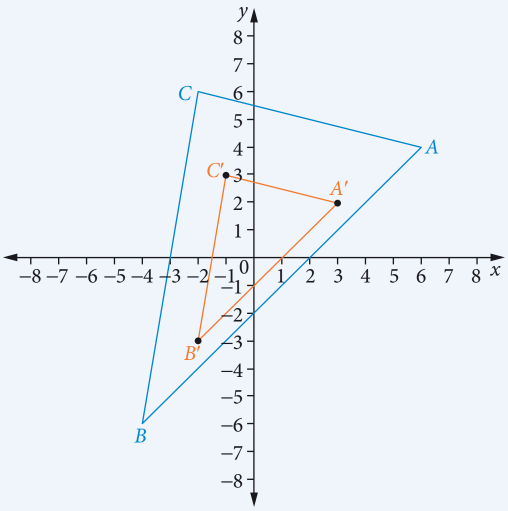
The dilation below reduces the triangle by a factor of 2 in both the x and y directions. The blue triangle is the original figure and the orange triangle is the reduced figure.



- What are the coordinates of A and A'?
- What are the coordinates of B and B'?
- How are the coordinates of the points for dilation by a factor of $\frac{1}{2}$ related?
- Complete the transformation for dilation by a factor of $\frac{1}{2}$ below.

  $(x, y) \rightarrow (\_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_)$
- What is the transformation for dilation by a factor of λ?

  $(x, y) \rightarrow (\_\_\_\_\_\_\_\_\_\_, \_\_\_\_\_\_\_\_\_\_)$

From the investigation, the following should be clear.

IMPORTANT

The transformation for **dilation** by a factor of λ_1 in the x direction and λ_2 in the y direction is given by $(x, y) \rightarrow (\lambda_1 x, \lambda_2 y)$.

For $\lambda_1, \lambda_2 > 1$, the dilation is a magnification that makes shapes larger. For $\lambda_1, \lambda_2 < 1$ the dilation is a reduction that makes shapes smaller.

Example 5

What is the transformation for dilation by a factor of 5 in the x direction and 3 in the y direction?

Solution

Write the general dilation.	$(x, y) \rightarrow (\lambda_1 x, \lambda_2 y)$
Substitute the factors $\lambda_1 = 5$ and $\lambda_2 = 3$.	$(x, y) \rightarrow (5x, 3y)$

Example 6

The parallelogram $A(2, -1)$ $B(4, 1)$ $C(1, 3)$ $D(-1, 1)$ is dilated by a factor of 2.5 in the x direction and 1.5 in the y direction.

a Find the image under the transformation.

b Show the object and image on common Cartesian axes.

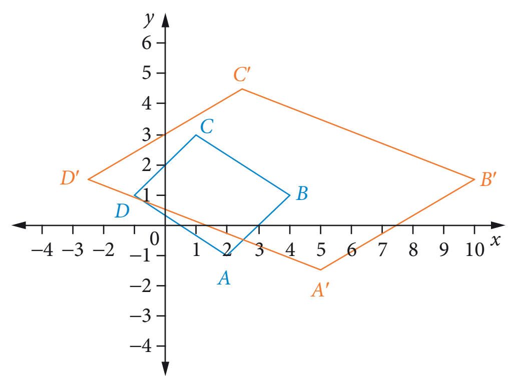
c What is the relationship between the shapes of the object and image?

Solution

a Apply the transformation $(x, y) \rightarrow (2.5x, 1.5y)$ to each point.

$A(2, -1) \rightarrow A'(5, -1.5)$
$B(4, 1) \rightarrow B'(10, 1.5)$
$C(1, 3) \rightarrow C'(2.5, 4.5)$
$D(-1, 1) \rightarrow D'(-2.5, 1.5)$

b Draw the object in blue and its image in orange.



c The object and image are both parallelograms, but the image is larger and distorted.

The object and image are the same kind of shape, but the image is larger and stretched more in the x direction.

A dilation by a factor of λ_1 and λ_2 in the x and y directions respectively can be written as $(x, y) \rightarrow (\lambda_1 x + 0y, 0x + \lambda_2 y)$ so it is a linear transformation. You can use the relationship between linear transformations and matrices to write the matrix for a dilation.

The matrix for the linear transformation

$(x, y) \rightarrow (ax + by, cx + dy)$ is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Substituting $a = \lambda_1$, $b = c = 0$ and $d = \lambda_2$ gives the following.

IMPORTANT

Dilation by factors of λ_1 and λ_2 in the x and y directions respectively is given by the matrix

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

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Example 7

a Write the matrix for a dilation by a factor of 0.6 in the x direction and 0.8 in the y direction.
b Use the matrix to find the image of the square $A(2, 3)$ $B(5, 8)$ $C(0, 11)$ $D(-3, 6)$.
c What is the shape of the image $A'B'C'D'$?

Solution

a Write the general matrix.

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Substitute $\lambda_1 = 0.6$ and $\lambda_2 = 0.8$.

$$\mathbf{D} = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}$$

b Apply the matrix to the position vector **a**.

$$\begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.2 \\ 2.4 \end{bmatrix}$$

Apply the matrix to the position vector **b**.

$$\begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 6.4 \end{bmatrix}$$

Apply the matrix to the position vector **c**.

$$\begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 0 \\ 8.8 \end{bmatrix}$$

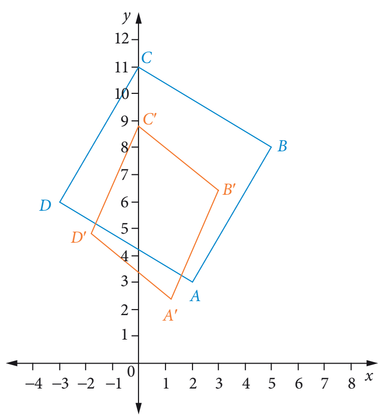
Apply the matrix to the position vector **d**.

$$\begin{bmatrix} 0.6 & 0 \\ 0 & 0.8 \end{bmatrix}\begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -1.8 \\ 4.8 \end{bmatrix}$$

Write the solution.

The image is $A'(1.2, 2.4)$ $B'(3, 6.4)$ $C'(0, 8.8)$ $D'(-1.8, 4.8)$.

c Sketch the graph.



The shape *appears* to be a rhombus. It is sufficient to prove that the diagonals bisect each other at right angles.
Find the midpoints of the diagonals and the slopes of the diagonals.

Midpoint of $A'C' = \left(\dfrac{1.2+0}{2}, \dfrac{2.4+8.8}{2}\right) = (0.6, 5.6)$

Midpoint of $B'D' = \left(\dfrac{3+(-1.8)}{2}, \dfrac{6.4+4.8}{2}\right) = (0.6, 5.6)$

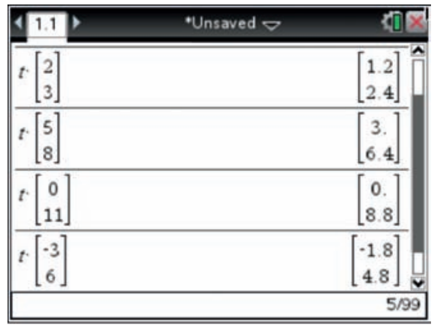
Slope of $A'C' = \dfrac{8.8-2.4}{0-1.2} = -\dfrac{16}{3}$

Slope of $B'D' = \dfrac{4.8-6.4}{(-1.8)-3} = \dfrac{1}{3}$

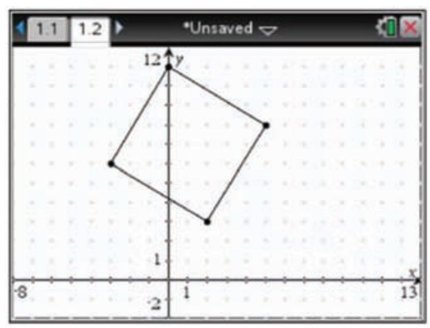
Use the midpoints.	The midpoints are the same so the diagonals bisect each other.
Use the slopes.	$m_1 m_2 = -\frac{16}{3} \times \frac{1}{3} = -\frac{16}{9} \neq -1$, so the diagonals are not perpendicular.
Make your conclusion.	The diagonals bisect each other, so the image $A'B'C'D'$ is a parallelogram but not a rhombus or square.

You can do this on your CAS calculator in two ways. You can define the matrix and apply it in the same way as for general linear transformations (see page s 424–425).

TI-Nspire CAS

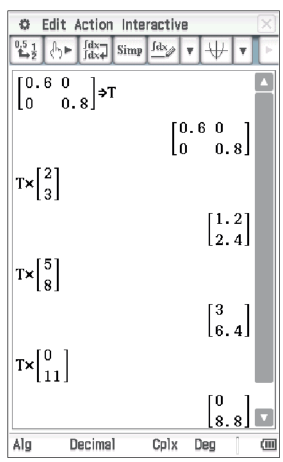


You can also add a Graph page and draw the shapes. Draw the square using 4: Polygon from the Shapes menu (see page 417) with Window settings of $-8 \leq x \leq 13$ and $-2 \leq y \leq 12$ for correct proportion.

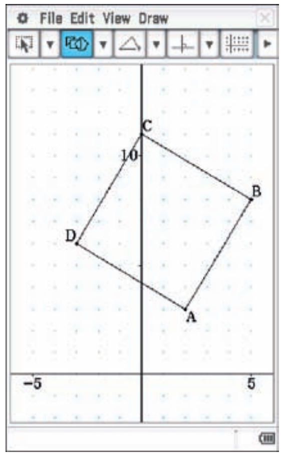


ClassPad

Ensure that your calculator is set to **Decimal**.



You can also use the **Geometry** application to draw the shapes.
Set View Window so $-6 \le x \le 6$ and ymid = 6 ($y = 6$ marks the middle of the screen.)
The easiest way to draw it is to tap the use the polygon tool ( ).



Notice that unless you carefully set the scales, the graph produced by the TI-Nspire is distorted so it is not a good guide to the real shape.

EXERCISE 11.03 Dilations

Concepts and techniques

1 Example 5 Write the transformations for each of the following dilations.
   a A dilation by a factor of 4 in the x direction and 3 in the y direction.
   b A dilation by a factor of 1.2 in the x direction and 1.2 in the y direction.
   c Magnification by a factor of 3 in the x direction and reduction by a factor of 2 in the y direction.
   d Reduction by a factor of 4 in the x direction and magnification by a factor of 3 in the y direction.

2 Example 7 a Write the matrix for a dilation by a factor of 1.5 in the x direction and 2 in the y direction.
   b Use the matrix to find the image of the triangle $A(3, 4)$ $B(4, 8)$ $C(7, 6)$.
   c What is the shape of the image?

3 a Write the matrix for a dilation by a factor of 0.4 in the x direction and 0.8 in the y direction.
   b Use the matrix to find the image of the rectangle $A(-10, -8)$ $B(-5, 4)$ $C(19, -6)$ $D(14, -18)$.
   c What is the shape of the image?

4 a CAS Write the matrix for a dilation by a factor of 0.7 in the x direction and 0.9 in the y direction.
   b Use the matrix to find the image of the square $A(-10, -8)$ $B(-4, 2)$ $C(6, -4)$ $D(0, -14)$.
   c What is the shape of the image?

5 a CAS Write the matrix for a dilation by a factor of 3 in the x direction and 2 in the y direction.
   b Use the matrix to find the image of the pentagon $A(-2, 1)$ $B(0, 4)$ $C(3, 4)$ $D(3, -1)$, $E(1, -2)$.
   c What is the shape of the image?

Reasoning and communication

6 Example 6 The triangle $A(-3, -6)$ $B(3, 3)$ $C(6, -3)$ is reduced by a factor of 3 in the x direction and 2 in the y direction.

a Find the image under the transformation.

b Show the object and image on the same Cartesian axes.

c What is the relationship between the object and image?

7 The square $A(-3, -5)$ $B(2, 7)$ $C(14, 2)$ $D(9, -10)$ is magnified by a factor of 2 in the x direction and 2.5 in the y direction.

a Find the image under the transformation.

b Show the object and image on the same Cartesian axes.

c What is the relationship between the object and image?

8 The parallelogram $A(-4, 4)$ $B(-2, 8)$ $C(4, 0)$ $D(2, -4)$ is dilated by a factor of 1.5 in both directions.

a Find the image under the transformation.

b Show the object and image on the same Cartesian axes.

c What is the relationship between the object and image?

11.04 ROTATIONS

A simple **rotation** turns all points through the same angle around the origin. The angle may be expressed in either radians or degrees. You need to use trigonometry to find the positions of the points that are rotated. Remember that angles are measured *anticlockwise* from the x-axis, so a rotation of 30° means an anticlockwise rotation of 30° and a rotation of $-\frac{5\pi}{4}$ means a rotation of $\frac{5\pi}{4}$ clockwise. Polar coordinates are easier than Cartesian coordinates to use with rotations. You studied polar coordinates in your vector work. The rules for changing between Cartesian and polar coordinates are given again below.

IMPORTANT

For a vector **v** with polar form (r, θ) and component form (x, y):

$x = r\cos(\theta)$ $\qquad$ $y = r\sin(\theta)$

$\tan(\theta) = \frac{y}{x}$ $\qquad$ $r^2 = x^2 + y^2$

and $|\mathbf{v}| = r = \sqrt{x^2 + y^2}$.

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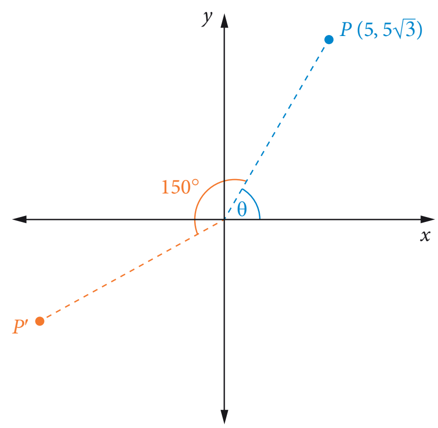
Example 8

Use trigonometry to find the image of the point $P(5, 5\sqrt{3})$ after a rotation of 150° around the origin.

Solution

Sketch a diagram.
Draw the point with an angle θ and the rotation through 150° to the new position P'.



Convert $P(5, 5\sqrt{3})$ to polar coordinates.

$r = \sqrt{5^2 + (5\sqrt{3})^2} = \sqrt{100} = 10$

$\tan(\theta) = \dfrac{5\sqrt{3}}{5} = \sqrt{3}$, so $\theta = 60°$

P is $P(10, 60°)$.

Add 150° to get the angle for P'.

P' is $P'(10, 210°)$.

Convert $P'(10, 210°)$ to Cartesian coordinates.

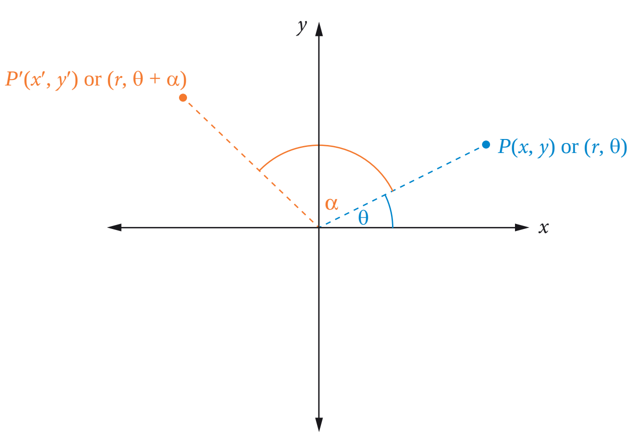
$x = r\cos(\theta) = 10\cos(210°) = -5\sqrt{3}$
$y = r\sin(\theta) = 10\sin(210°) = -5$

Write the answer.

The image is $P'(-5\sqrt{3}, -5)$.

Calculation of each point as shown in Example **8** is time-consuming. You can use the same approach to find the general form of rotations about the origin.

Consider a rotation of the point (x, y) through an angle of α. You can draw a diagram similar to that shown in Example **8**.



The polar coordinates of P' are $(r, \theta + \alpha)$, so the Cartesian coordinates of P' are given by

$$x' = r\cos(\theta + \alpha) \text{ and } y' = r\sin(\theta + \alpha)$$

Using the identities

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

and

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

gives

$$x' = r[\cos(\theta)\cos(\alpha) - \sin(\theta)\sin(\alpha)]$$

and

$$y' = r[\sin(\theta)\cos(\alpha) + \cos(\theta)\sin(\alpha)].$$

Expanding, rearranging the products and using $a + b = b + a$ with y' gives

$$x' = [r\cos(\theta)]\cos(\alpha) - [r\sin(\theta)]\sin(\alpha)$$

and

$$y' = [r\cos(\theta)]\sin(\alpha) + [r\sin(\theta)]\cos(\alpha)$$

But the Cartesian coordinates of P are $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Substituting gives

$$x' = x\cos(\alpha) - y\sin(\alpha) \text{ and } y' = x\sin(\alpha) + y\cos(\alpha)$$

This gives the rule for rotation through an angle α around the origin as follows.

IMPORTANT

The transformation for **rotation** through an angle α around the origin is given by

$$(x, y) \to (x\cos(\alpha) - y\sin(\alpha),\ x\sin(\alpha) + y\cos(\alpha))$$

where α is the usual anticlockwise rotation. For a clockwise rotation, α is negative.

Example 9

Use the transformation to find the image of the triangle $A(2, 2)$ $B(5, 6)$ $C(6, -1)$ after rotation through an angle of 125°, correct to 2 decimal places.

Solution

Write the general rotation.	$(x, y) \to (x\cos(\alpha) - y\sin(\alpha),\ x\sin(\alpha) + y\cos(\alpha))$
Substitute $\alpha = 125°$.	$(x, y) \to (x\cos(125°) - y\sin(125°),\ x\sin(125°) + y\cos(125°))$
Apply to $A(2, 2)$.	$A' = (2\cos(125°) - 2\sin(125°),\ 2\sin(125°) + 2\cos(125°))$
Simplify and round.	$\approx (-2.79, 0.49)$
Apply to $B(5, 6)$.	$B' = (5\cos(125°) - 6\sin(125°),\ 5\sin(125°) + 6\cos(125°))$
Simplify and round.	$\approx (-7.78, 0.65)$
Apply to $C(6, -1)$.	$C' = (6\cos(125°) - (-1)\sin(125°),\ 6\sin(125°) + (-1)\cos(125°))$
Simplify and round.	$\approx (-2.62, 5.49)$
Write the solution.	The image is $A'(-2.79, 0.49)$ $B'(-7.78, 0.65)$ $C'(-2.62, 5.49)$.

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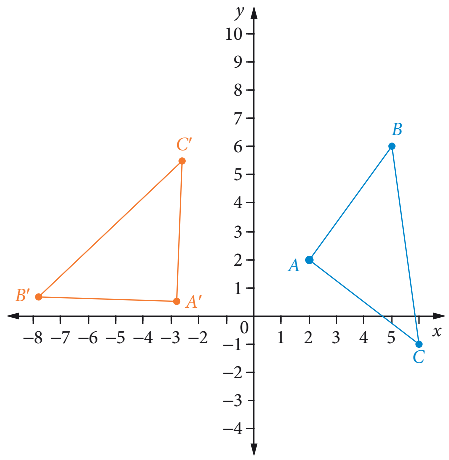
The diagram below shows the object and image from Example **9**. As you would expect, the rotation preserves the shape so that the image is **congruent** to the object.



The matrix for the linear transformation $(x, y) \rightarrow (ax + by, cx + dy)$ is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
Substituting $a = d = \cos(\alpha)$, $b = -\sin(\alpha)$ and $c = \sin(\alpha)$ gives the following.

IMPORTANT

Rotation through an angle α around the origin is given by the matrix

$$\mathbf{R}_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

Unless otherwise directed, use exact values for rotations whenever possible.

Example 10

a What is the matrix for a rotation of $-\frac{7\pi}{6}$ around the origin?

b Use the matrix to find the image of the rectangle $A(-2, -3)$ $B(1, 1)$ $C(9, -5)$ $D(6, -9)$ after a rotation of $-\frac{7\pi}{6}$.

Solution

a Write the general formula.

$$\mathbf{R}_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

Substitute values.

$$\mathbf{R}_{-\frac{7\pi}{6}} = \begin{bmatrix} \cos\left(-\frac{7\pi}{6}\right) & -\sin\left(-\frac{7\pi}{6}\right) \\ \sin\left(-\frac{7\pi}{6}\right) & \cos\left(-\frac{7\pi}{6}\right) \end{bmatrix}$$

Simplify.

$$= \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

b Apply the matrix to A.

$$\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix}=\begin{bmatrix} \frac{3+2\sqrt{3}}{2} \\ \frac{3\sqrt{3}-2}{2} \end{bmatrix}$$

Apply the matrix to B.

$$\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} -\frac{1+\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{bmatrix}$$

Apply the matrix to C.

$$\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 9 \\ -5 \end{bmatrix}=\begin{bmatrix} \frac{5-9\sqrt{3}}{2} \\ \frac{9+5\sqrt{3}}{2} \end{bmatrix}$$

Apply the matrix to D.

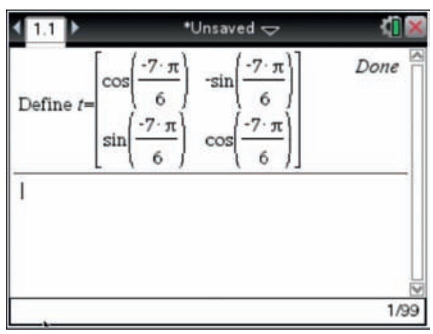
$$\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 6 \\ -9 \end{bmatrix}=\begin{bmatrix} \frac{9-6\sqrt{3}}{2} \\ \frac{6+9\sqrt{3}}{2} \end{bmatrix}$$

Write the solution.

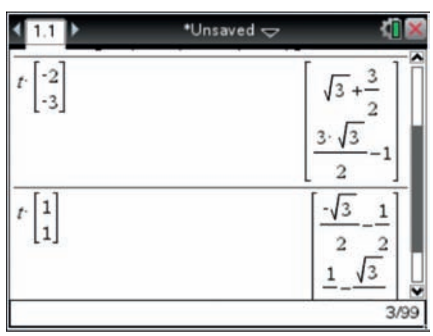
The image is $A'\left(\frac{3+2\sqrt{3}}{2}, \frac{3\sqrt{3}-2}{2}\right)$ $B'\left(-\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}\right)$ $C'\left(\frac{5-9\sqrt{3}}{2}, \frac{9+5\sqrt{3}}{2}\right)$ $D'\left(\frac{9-6\sqrt{3}}{2}, \frac{6+9\sqrt{3}}{2}\right)$.

TI-Nspire CAS

You can do this on your CAS calculator in the same way as for general linear transformations (see pages 424–425). Make sure that you set your calculator in radians for this problem.



Calculate each position vector in succession. The calculator separates the rational and irrational parts of the answers.



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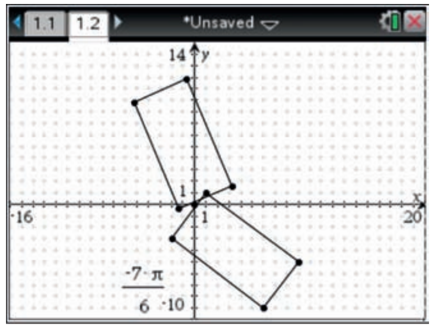
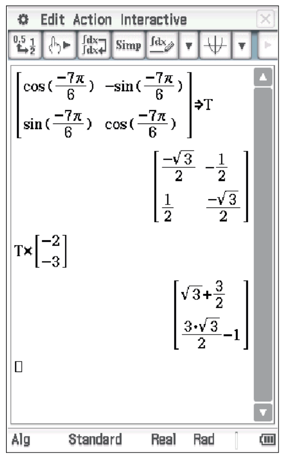
Also, you can add a Graph page and draw the rectangle (see page 417). Make the Window settings $-16 \le x \le 20$ and $-10 \le y \le 20$.
Then use menu, 8: Geometry, 5: Transformation and 4: Rotation to rotate the rectangle by clicking on the origin, clicking on the rectangle and entering the angle.



ClassPad

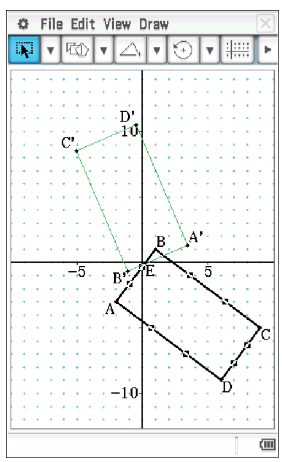
You can define the matrix and calculate the image points as before (see pages 424–425). The multiplication sign is essential.
Only the first point has been calculated in the screen on the right.

The calculator should be set to **Standard** and **Rad**.



You can also use the **Geometry** application. Set the View Window to $-10 \le x \le 10$ and ymid to 1 and tap **OK**. Draw the rectangle (see page 418). Make sure the **Function Angle** and **Measure Angle** are set to radians by tapping and selecting **Geometry Format**.

Now tap the selection tool, select all sides and tap **Draw**, **Construct** and **Rotation**. Tap on the origin as the centre of rotation and enter the angle.



EXERCISE 11.04 Rotations

Concepts and techniques

WS Rotations

1 Example 8 Use trigonometry to find the image of $P(6, 6)$ after a rotation of 210° around the origin.

2 Use trigonometry to find the image of $P(-2\sqrt{3}, 2)$ after a rotation of $\frac{2\pi}{3}$ around the origin.

3 Example 9 Find the image of the quadrilateral $A(2, 2)$ $B(5, 6)$ $C(6, -1)$ $D(-2, -2)$ after a rotation through an angle of 208°, correct to 2 decimal places.

4 Find the image of the kite $A(3, 7)$ $B(8, 6)$ $C(7, 1)$ $D(-4, -2)$ after a rotation through an angle of 208°, correct to 2 decimal places.

5 Example 10 **a** What is the matrix for a rotation of 323° around the origin, correct to 2 decimal places?

b Use the matrix to find the image of the triangle $A(4, 5)$ $B(6, 8)$ $C(7, -1)$ after a rotation of 323°, correct to 2 decimal places.

6 **a** CAS What is the matrix for a rotation of 137° around the origin, correct to 2 decimal places?

b Use the matrix to find the image of the pentagon $A(-4, 2)$ $B(-2, 6)$ $C(4, 8)$ $D(5, 6)$ $E(2, 6)$ after a rotation of 137°, correct to 2 decimal places.

7 **a** CAS What is the matrix for a rotation of $\frac{3\pi}{4}$ around the origin?

b Use the matrix to find the image of the parallelogram $A(-2, -2)$ $B(1, 3)$ $C(9, 3)$ $D(6, -2)$ after a rotation of $\frac{3\pi}{4}$.

Reasoning and communication

8 **a** What is the matrix for a rotation of 60°?

b What rotation would restore a figure transformed by the rotation in part **a** to its original position?

c What is the matrix for the restoring rotation?

9 **a** What is the matrix for a rotation of $\frac{7\pi}{4}$?

b What rotation would restore a figure transformed by the rotation in part **a**?

c What is the matrix for the restoring rotation?

10 The matrix for a rotation is $\mathbf{R}_\theta = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

a What is the angle of rotation for the transformation?

b What is the matrix for the rotation that would restore a figure transformed by this matrix to its original position?

11.05 REFLECTIONS

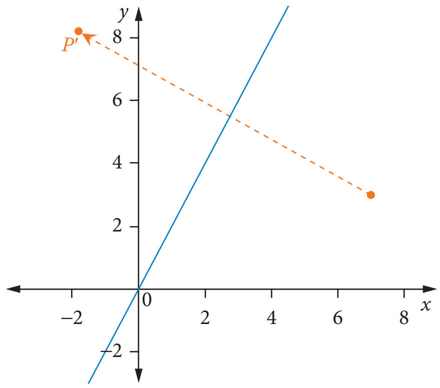
A simple **reflection** swaps all points across to the opposite side of a line through the origin. The line may be expressed by its inclination angle, gradient or equation. You can use your knowledge of coordinate geometry to find the positions of points that are reflected. Remember that the equation of a straight line with gradient m and y-intercept c is given by $y = mx + c$. The gradient of a line with inclination θ is given by $m = \tan(\theta)$.

Example 11

Find the image of the point $P(7, 3)$ when it is reflected across the line $y = 2x$.

Solution

Sketch a diagram.
Point P' will be the same distance as P from the line, but on the other side.
The line PP' is perpendicular to the line $y = 2x$ and passes through point P.



Find the slope of PP'.	The slope of $y = 2x$ is 2, so PP' has slope $-\frac{1}{2}$.
Find the equation of PP' using $y - y_1 = m(x - x_1)$	$y - 3 = -\frac{1}{2}(x - 7)$ $x + 2y - 13 = 0$
Substitute $y = 2x$ to find the intersection.	$x + 2 \times 2x - 13 = 0$
Find x.	$x = 2.6$
Substitute in $y = 2x$ to find y.	$y = 2 \times 2.6 = 5.2$
State the intersection.	$y = 2x$ and PP' intersect at $M(2.6, 5.2)$.
State the relationship of P, M and P'.	M is the midpoint of PP'.
Use the midpoint formula, $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.	$M(2.6, 5.2) = \left(\frac{7+x_2}{2}, \frac{3+y_2}{2}\right)$, where (x_2, y_2) are the coordinates of P'.
Find x_2.	$\frac{7+x_2}{2} = 2.6$ $x_2 + 7 = 5.2$ $x_2 = -1.8$
Find y_2.	$\frac{3+y_2}{2} = 5.2$ $y_2 + 3 = 10.4$ $y_2 = 7.4$
Write the answer.	The image is $P'(-1.8, 7.4)$.

Repeating the procedure in Example **11** for many points to find the reflection of a geometric shape would take a very, very long time. You can use the same procedure as shown in Example **11** to find the general transformation for reflection in a line through the origin.

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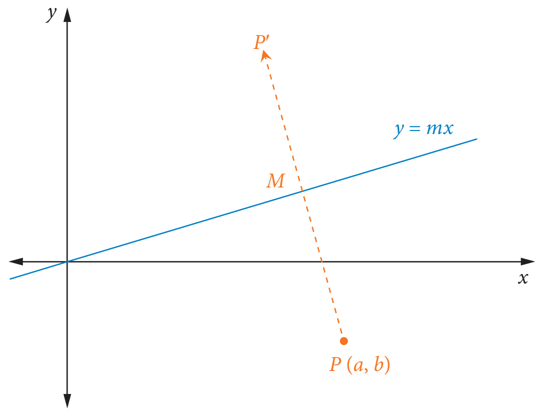
Consider the reflection of a point across the line $y = mx$.

Sketch a diagram similar to that in the example.



The slope of the line of reflection is m so the slope of PP' is $-\frac{1}{m}$.

The equation of PP' is given by $y - b = -\frac{1}{m}(x - a)$. Substituting $y = 2x$ gives

$2x - b = -\frac{1}{m}(x - a)$ giving $x = \frac{a+mb}{1+m^2}$.

Substituting in $y = mx$ gives the coordinates of M as $\left(\frac{a+mb}{1+m^2}, \frac{ma+m^2b}{1+m^2}\right)$.

M is the midpoint of $P(a, b)$ and $P'(c, d)$, so $\left(\frac{a+mb}{1+m^2}, \frac{ma+m^2b}{1+m^2}\right) = \left(\frac{a+c}{2}, \frac{b+d}{2}\right)$.

To find c, you solve $\frac{a+mb}{1+m^2} = \frac{a+c}{2}$ to get $c = \left(\frac{1-m^2}{1+m^2}\right)a + \left(\frac{2m}{1+m^2}\right)b$.

To find d, you solve $\frac{ma+m^2b}{1+m^2} = \frac{b+d}{2}$ to get $d = \left(\frac{2m}{1+m^2}\right)a - \left(\frac{1-m^2}{1+m^2}\right)b$

Replacing (a, b) by (x, y) and (c, d) by (x', y') gives

$x' = px + qy$ and $y' = qx - py$ where $p = \frac{1-m^2}{1+m^2}$ and $q = \frac{2m}{1+m^2}$.

If you know the inclination of the line, you can use that directly to find the values of p and q by substituting $m = \tan(\theta)$ in the equations for p and q.

$$p = \frac{1-\tan^2(\theta)}{1+\tan^2(\theta)} = \frac{1-\frac{\sin^2(\theta)}{\cos^2(\theta)}}{1+\frac{\sin^2(\theta)}{\cos^2(\theta)}} = \frac{\cos^2(\theta)-\sin^2(\theta)}{\cos^2(\theta)+\sin^2(\theta)} = \cos(2\theta)$$

and

$$q = \frac{2\tan(\theta)}{1+\tan^2(\theta)} = \frac{\frac{2\sin(\theta)}{\cos(\theta)}}{1+\frac{\sin^2(\theta)}{\cos^2(\theta)}} = \frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta)+\sin^2(\theta)} = \sin(2\theta)$$

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You can substitute in the general linear transformation matrix to get the matrix for reflection.

IMPORTANT

The transformation for **reflection** across a line with slope m through the origin is given by

$$(x, y) \to \left(\left(\frac{1-m^2}{1+m^2}\right)x + \left(\frac{2m}{1+m^2}\right)y, \left(\frac{2m}{1+m^2}\right)x - \left(\frac{1-m^2}{1+m^2}\right)y\right)$$

or $(x, y) \to (px + qy, qx - py)$, where $p = \frac{1-m^2}{1+m^2}$ and $q = \frac{2m}{1+m^2}$.

The transformation for **reflection** across a line of inclination θ through the origin is given by

$$(x, y) \to (\cos(2\theta)x + \sin(2\theta)y, \sin(2\theta)x - \cos(2\theta)y)$$

The matrix for **reflection** through a line with a slope of m or inclination $\theta = \tan^{-1}(m)$ through the origin is given by the matrix

$$\mathbf{M} = \begin{bmatrix} p & q \\ q & -p \end{bmatrix} \text{ or } \mathbf{M} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

where $p = \frac{1-m^2}{1+m^2}$ and $q = \frac{2m}{1+m^2}$

You should use the first method if you are given the slope of the line of reflection and the second if you are given the angle of inclination.

Example 12

Use the transformations given below to find the image of the triangle $A(-5, -5)$ $B(5, 0)$ $C(0, -10)$

a after reflection across a line through the origin with slope -0.5

b after reflection across a line through the origin with inclination $\frac{\pi}{6}$ to the x-axis.

Solution

a Find the value of p.

$$p = \frac{1-m^2}{1+m^2} = \frac{1-0.25}{1+0.25} = 0.6$$

Find the value of q.

$$q = \frac{2m}{1+m^2} = \frac{2\times(-0.5)}{1+0.25} = -0.8$$

Write the matrix for the reflection.

$$\mathbf{M} = \begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix}$$

Apply the transformation to ABC using $\begin{bmatrix} -5 & 5 & 0 \\ -5 & 0 & -10 \end{bmatrix}$ to represent A, B and C.

$$\begin{bmatrix} 0.6 & -0.8 \\ -0.8 & -0.6 \end{bmatrix}\begin{bmatrix} -5 & 5 & 0 \\ -5 & 0 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 8 \\ 7 & -4 & 6 \end{bmatrix}$$

Write the answer.

The image is $A'(1, 7)$ $B'(3, -4)$ $C'(8, 6)$.

b Find cos (2θ).

$$\cos\left(2\times\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$$

Find sin (2θ).

$$\sin\left(2\times\frac{\pi}{6}\right)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

Write the matrix for **M**.

$$\mathbf{M}=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}$$

Apply the transformation to ABC.

$$\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}-5 & 5 & 0\\ -5 & 0 & -10\end{bmatrix}$$

Work out the answer for $A'B'C'$.

$$=\begin{bmatrix}\frac{-5\left(1+\sqrt{3}\right)}{2} & \frac{5}{2} & -5\sqrt{3}\\ \frac{5\left(1-\sqrt{3}\right)}{2} & \frac{5\sqrt{3}}{2} & 5\end{bmatrix}$$

Write the answer.

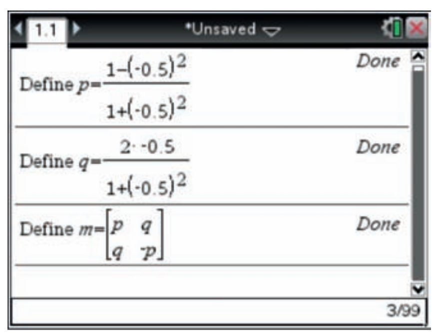
The image is $A'\left(\frac{-5\left(1+\sqrt{3}\right)}{2},\frac{5\left(1-\sqrt{3}\right)}{2}\right)$

$B'\left(\frac{5}{2},\frac{5\sqrt{3}}{2}\right)$ $C'\left(-5\sqrt{3},5\right)$.

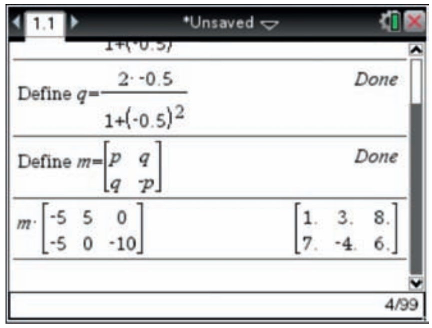
You can use your CAS calculator to perform the matrix calculation.

TI-Nspire CAS

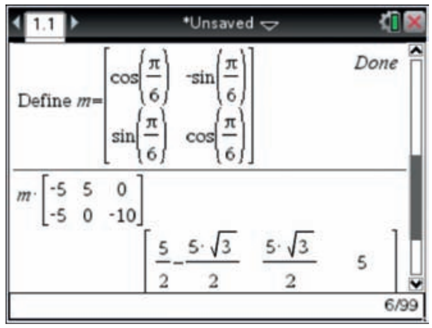
a Define the matrix first.



Then perform the calculations for the triangle ABC.

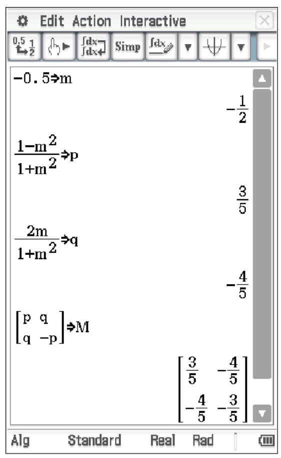


b In this case, make sure that your calculator is set on radians before defining the matrix.

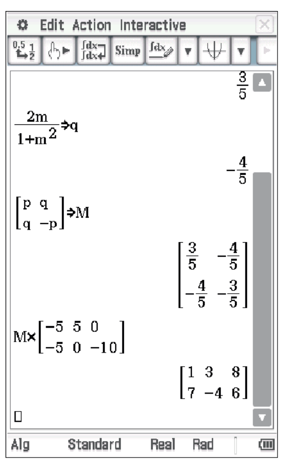


ClassPad

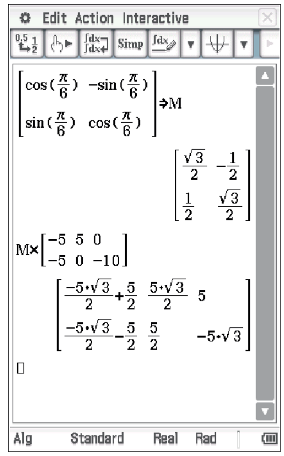
a Use the $\sqrt{\alpha}$ Main application.
Ensure that the calculator is set to Standard.
First define m, then define p and q, and finally define the matrix M.



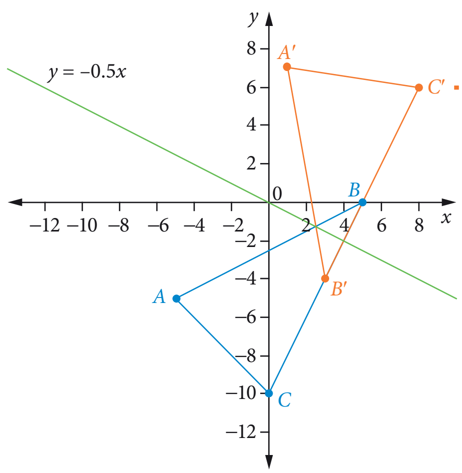
Perform the calculation for ΔABC.
Press Keyboard and tap Math2.
Use [▦] to open a 2×2 matrix.
Widen it to 2×3 by tapping [▥]. Enter the elements of the matrix. Each column corresponds to one point.



b In this case, make sure that your calculator is set to radians.



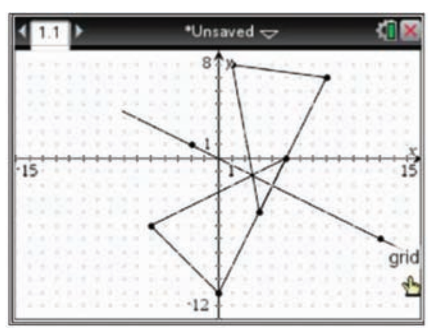
The reflection for Example **12** (part **a**) is shown below. Notice that the shape of the figure is preserved. The image $A'B'C'$ is **congruent** to the object ABC.



You can also use your CAS calculator to draw the triangles and reflect them without using a matrix, but you have to draw the line first.

TI-Nspire CAS

Use menu, 8: Geometry, 5: Transformation and 2: Reflection to reflect the triangle by clicking on it and clicking on the line.

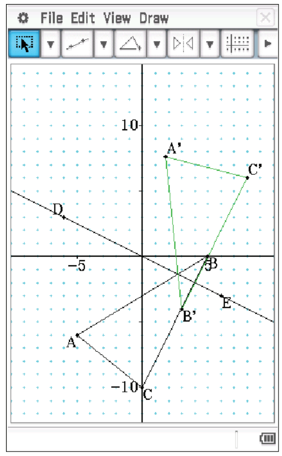


ClassPad

Select the **Geometry** application.
Draw the triangle ABC using the **Line Segment** command.

To draw the line of reflection, select the Line and on the screen tap at one point that is on the line y = –0.5x. Drag the stylus to a second point on the line to draw a line from D to E.

Tap and select each side of the triangle. Tap **Draw**, **Construct**, **Reflection**. Tap the line.



If you recognise the inclination angle from the slope, it is better to use the angle form of the reflection transformation because it is easier to apply.

EXERCISE 11.05 Reflections

Concepts and techniques

WS Reflections

1 Example 11 Use coordinate geometry to find the following.
   a The image of $P(5, 10)$ when it is reflected in the line $y = -2x$
   b The image of $P(3, 7)$ when it is reflected in the line $y = x$
   c The image of $P(-9, 3)$ when it is reflected in the line $y = 3x$
   d The image of $P(20, -15)$ when it is reflected in the line $y = \frac{x}{7}$

2 Example 12 Use a matrix to find the image of $A(-2, 1)$ $B(-1, 3)$ $C(-2, -2)$ after it is reflected across a line through the origin with slope 2.

3 Use a matrix to find the image of $A(5, 5)$ $B(-5, 0)$ $C(-1, 6)$ after it is reflected across a line through the origin with slope $-\frac{1}{3}$.

4 CAS Use a matrix to find the image of $A(-10, 10)$ $B(-5, -5)$ $C(5, -15)$ $D(10, -5)$ after it is reflected across a line through the origin with slope 7.

5 CAS Use a matrix to find the image of $A(4, 5)$ $B(7, 9)$ $C(11, 6)$ $D(7, 1)$ $E(5, 0)$ after it is reflected across a line through the origin with slope –3.

6 Use a matrix to find the image of $A(4, 5)$ $B(6, 9)$ $C(3, 7)$ after it is reflected across a line through the origin with inclination 30°, correct to 2 decimal places.

7 Use a matrix to find the image of $A(-4, 5)$ $B(5, 15)$ $C(0, 20)$ after it is reflected across a line through the origin with inclination $\frac{3\pi}{8}$, correct to 2 decimal places.

8 CAS Use a matrix to find the image of $A(2, -5)$ $B(7, -1)$ $C(15, -7)$ $D(12, -11)$ after it is reflected across a line through the origin with inclination 75°, correct to 2 decimal places.

9 CAS Use a matrix to find the image of $A(1, 5)$ $B(6, 9)$ $C(14, 3)$ $D(11, -1)$ $E(9, 4)$ after it is reflected across a line through the origin with inclination $-\frac{\pi}{15}$, correct to 2 decimal places.

Reasoning and communication

10 Use matrices or other methods to show that any shape is restored to its original position after a reflection in a line through the origin by the same reflection.

11.06 COMPOSITION OF TRANSFORMATIONS

In mathematics, the term **composition** means the application of two processes of the same type, one after the other. Composition is shown by the symbol $\circ$, so $\mathbf{S} \circ \mathbf{T}$ means 'the composition of **S** and **T**'. The order of application is from right to left, so **T** is performed first, then **S** is applied to the result of **T**.

IMPORTANT

The composition of the transformations **S** and **T** is the combined transformation such that if $\mathbf{T}: (x, y) \to (x', y')$ and $\mathbf{S}: (x', y') \to (x'', y'')$, then the composition is

$\mathbf{S} \circ \mathbf{T}: (x, y) \to (x', y') \to (x'', y'')$.

The image of a vector **v** under a transformation **T** is written as **Tv**.

Example 13

The transformation **S** is a translation 4 to the left and 5 up. **T** is a translation 3 to the right and 2 up.

a Write the translations in the form $(x, y) \to (x', y')$ and find the composition $\mathbf{S} \circ \mathbf{T}$.

b Write the translations as column vectors and find the composition $\mathbf{S} \circ \mathbf{T}$.

c Find the image of $A(2, 2)$ $B(6, 5)$ $C(9, 2)$ $D(7, -2)$ under $\mathbf{S} \circ \mathbf{T}$.

d Draw the object and image from part **c** on the same set of axes.

Solution

a **S** decreases x by 4 and increases y by 5.	$\mathbf{S}: (x, y) \to (x - 4, y + 5)$
T increases x by 3 and increases y by 2.	$\mathbf{T}: (x, y) \to (x + 3, y + 2)$
Apply **S** to the image from **T**.	$\mathbf{S}: ((x + 3), (y + 2)) \to ((x + 3) - 4, (y + 2) + 5)$
Simplify.	$((x + 3) - 4, (y + 2) + 5) = (x - 1, y + 7)$
Write the result for $\mathbf{S} \circ \mathbf{T}$.	$\mathbf{S} \circ \mathbf{T}: (x, y) \to (x - 1, y + 7)$
b Write **S** as a column matrix.	$\mathbf{S} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$
Write **T** as a column matrix.	$\mathbf{T} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

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Find $\mathbf{Tv}$ for $\mathbf{v}=\begin{bmatrix}x\\y\end{bmatrix}$ by addition.

$\mathbf{Tv}=\mathbf{T}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}+\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3+x\\2+y\end{bmatrix}$

Now find $\mathbf{S}(\mathbf{Tv})$.

$$\begin{aligned}\mathbf{S}(\mathbf{Tv})&=\mathbf{S}\begin{bmatrix}3+x\\2+y\end{bmatrix}\\&=\begin{bmatrix}-4\\5\end{bmatrix}+\begin{bmatrix}3+x\\2+y\end{bmatrix}\\&=\begin{bmatrix}-4+3+x\\5+2+y\end{bmatrix}\\&=\begin{bmatrix}-1+x\\7+y\end{bmatrix}\end{aligned}$$

Write the result for $(\mathbf{S}\circ\mathbf{T})\mathbf{v}$.

$(\mathbf{S}\circ\mathbf{T})\mathbf{v}=\mathbf{S}(\mathbf{Tv})=\begin{bmatrix}-1+x\\7+y\end{bmatrix}=\begin{bmatrix}-1\\7\end{bmatrix}+\begin{bmatrix}x\\y\end{bmatrix}$

Write the column matrix for $\mathbf{S}\circ\mathbf{T}$.

$\mathbf{S}\circ\mathbf{T}=\begin{bmatrix}-1\\7\end{bmatrix}$

c Apply $\mathbf{S}\circ\mathbf{T}$ to the position vector of A.

$(\mathbf{S}\circ\mathbf{T})\mathbf{a}=\begin{bmatrix}-1\\7\end{bmatrix}+\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}1\\9\end{bmatrix}$

Apply $\mathbf{S}\circ\mathbf{T}$ to the position vector of B.

$(\mathbf{S}\circ\mathbf{T})\mathbf{b}=\begin{bmatrix}-1\\7\end{bmatrix}+\begin{bmatrix}6\\5\end{bmatrix}=\begin{bmatrix}5\\12\end{bmatrix}$

Apply $\mathbf{S}\circ\mathbf{T}$ to the position vector of C.

$(\mathbf{S}\circ\mathbf{T})\mathbf{c}=\begin{bmatrix}-1\\7\end{bmatrix}+\begin{bmatrix}9\\2\end{bmatrix}=\begin{bmatrix}8\\9\end{bmatrix}$

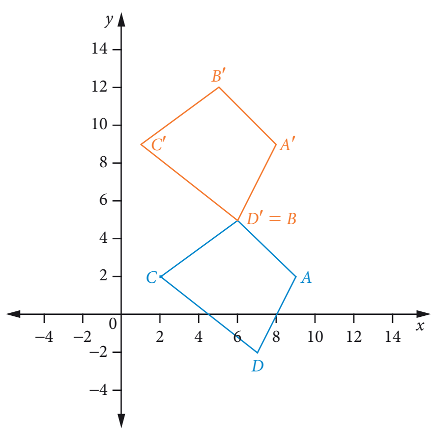
Apply $\mathbf{S}\circ\mathbf{T}$ to the position vector of D.

$(\mathbf{S}\circ\mathbf{T})\mathbf{d}=\begin{bmatrix}-1\\7\end{bmatrix}+\begin{bmatrix}7\\-2\end{bmatrix}=\begin{bmatrix}6\\5\end{bmatrix}$

Write the result.

The image is $A'(1, 9)$ $B'(5, 12)$ $C'(8, 9)$ $D'(6, 5)$.

d Show the object and image on the same axes in different colours.



What kind of transformation does the composition of two translations produce?

Dilations, rotations and reflections are all linear transformations. It is possible to show that the composition of two linear transformations is also a linear transformation. You can use the product of the matrices to find the effect of composing two linear transformations.

Example 14

The rotation though an angle of 60° is written as $\mathbf{R}_{60°}$ and the dilation by a factor of 2 in the x direction and 4 in the y direction is written as $\mathbf{D}$.

a Use matrices to find the transformation $\mathbf{D} \circ \mathbf{R}_{60°}$

b Use matrices to find the transformation $\mathbf{R}_{60°} \circ \mathbf{D}$.

c Is this composition commutative?

d Find the image of $A(3, 6)$ $B(5, 4)$ $C(8, 5)$ under $\mathbf{R}_{60°} \circ \mathbf{D}$.

e Show ABC and $A'B'C'$ on the same set of axes.

Solution

a Write the matrix for $\mathbf{D}$.

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

Write the matrix for $\mathbf{R}_{60°}$.

$$\mathbf{R}_{60°} = \begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Work out $\mathbf{D} \times \mathbf{R}_{60°}$.

$$\mathbf{D} \circ \mathbf{R}_{60°} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{3} \\ 2\sqrt{3} & 2 \end{bmatrix}$$

b Work out $\mathbf{R}_{60°} \times \mathbf{D}$.

$$\mathbf{R}_{60°} \circ \mathbf{D} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -2\sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix}$$

c You cannot reverse the order.

Since $\mathbf{R}_{60°} \circ \mathbf{D} \neq \mathbf{D} \circ \mathbf{R}_{60°}$, this composition is not commutative.

d Apply $\mathbf{R}_{60°} \circ \mathbf{D}$ to ABC.

$$\begin{bmatrix} 1 & -2\sqrt{3} \\ \sqrt{3} & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 & 8 \\ 6 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 3-12\sqrt{3} & 5-8\sqrt{3} & 8-10\sqrt{3} \\ 12+3\sqrt{3} & 8+5\sqrt{3} & 10+8\sqrt{3} \end{bmatrix}$$

Write the answer.

The image is $A'(3 - 12\sqrt{3}, 12 + 3\sqrt{3})$ $B'(5 - 8\sqrt{3}, 8 + 5\sqrt{3})$ $C'(8 - 10\sqrt{3}, 10 + 8\sqrt{3})$.

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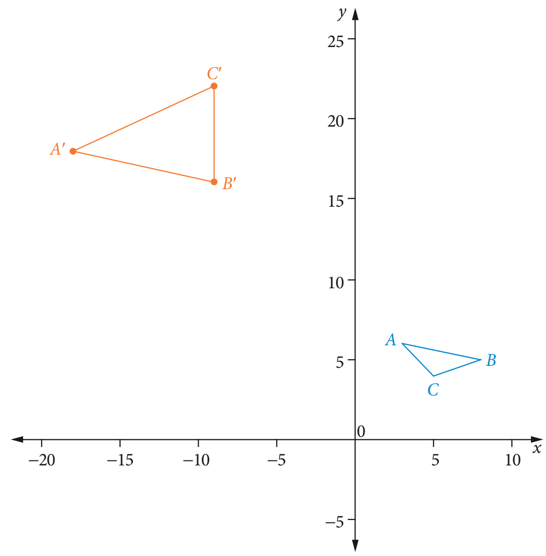
e Use approximations to sketch the triangles on the same axes.



Example 15

The transformation **M** is a reflection in the line through the origin with slope 3, and the transformation $\mathbf{R}_{120°}$ is a rotation through an angle of 120°.

a Use matrices to find the composition $\mathbf{M} \circ \mathbf{R}_{120°}$.

b Use matrices to find the composition $\mathbf{R}_{120°} \circ \mathbf{M}$.

c Is the composition commutative?

d Find the image of $A(2, -2)$ $B(4, 3)$ $C(5, -1)$ under $\mathbf{R}_{120°} \circ \mathbf{M}$.

e Show ABC and $A'B'C'$ on the same set of axes.

Solution

a Find p.

$$p = \frac{1-m^2}{1+m^2} = \frac{1-9}{1+9} = -\frac{4}{5} = -0.8$$

Find q.

$$q = \frac{2m}{1+m^2} = \frac{2 \times 3}{1+9} = \frac{3}{5} = 0.6$$

Write **M**.

$$\mathbf{M} = \begin{bmatrix} p & q \\ q & -p \end{bmatrix} = \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$$

Write $\mathbf{R}_{120°}$.

$$\mathbf{R}_{120°} = \begin{bmatrix} \cos(120°) & -\sin(120°) \\ \sin(120°) & \cos(120°) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

Find $\mathbf{M} \circ \mathbf{R}_{120^\circ}$.

$$\mathbf{M} \circ \mathbf{R}_{120^\circ} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3\sqrt{3}+4}{10} & \frac{4\sqrt{3}-3}{10} \\ \frac{4\sqrt{3}-3}{10} & \frac{-(3\sqrt{3}+4)}{10} \end{bmatrix}$$

b Find $\mathbf{R}_{120^\circ} \circ \mathbf{M}$.

$$\mathbf{R}_{120^\circ} \circ \mathbf{M} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4-3\sqrt{3}}{10} & \frac{-(4\sqrt{3}+3)}{10} \\ \frac{-(4\sqrt{3}+3)}{10} & \frac{3\sqrt{3}-4}{10} \end{bmatrix}$$

c Compare the answers for parts **a** and **b**.

$\mathbf{M} \circ \mathbf{R}_{120^\circ} \neq \mathbf{R}_{120^\circ} \circ \mathbf{M}$ so the composition is not commutative.

d Apply $\mathbf{R}_{120^\circ} \circ \mathbf{M}$ to ABC.

$$\begin{bmatrix} \frac{4-3\sqrt{3}}{10} & \frac{-(4\sqrt{3}+3)}{10} \\ \frac{-(4\sqrt{3}+3)}{10} & \frac{3\sqrt{3}-4}{10} \end{bmatrix} \begin{bmatrix} 2 & 4 & 5 \\ -2 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7+2\sqrt{3}}{5} & \frac{7-24\sqrt{3}}{10} & \frac{23-11\sqrt{3}}{10} \\ \frac{(1-7\sqrt{3})}{5} & \frac{-(24+7\sqrt{3})}{10} & \frac{-(11+23\sqrt{3})}{5} \end{bmatrix}$$

$$\approx \begin{bmatrix} 1.74\ldots & -3.45\ldots & 0.39\ldots \\ -2.22\ldots & -3.61\ldots & -5.08\ldots \end{bmatrix}$$

Write the answer.

The image is $A'\left(\frac{7+\sqrt{3}}{5}, \frac{(1-7\sqrt{3})}{5}\right)$

$B'\left(\frac{7-24\sqrt{3}}{10}, \frac{-(24+7\sqrt{3})}{10}\right)$

$C'\left(\frac{23-11\sqrt{3}}{10}, \frac{-(11+23\sqrt{3})}{5}\right)$.

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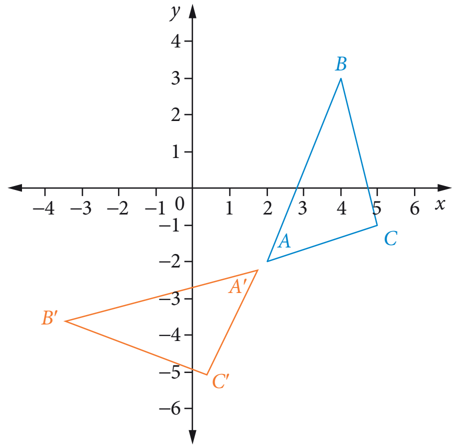
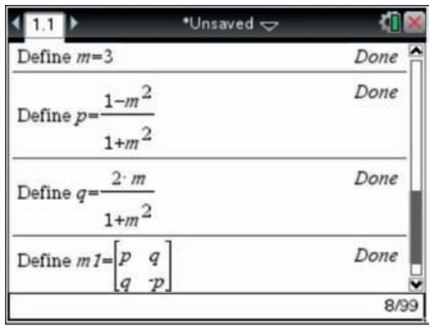
e Use approximations to sketch the triangles on the same axes.

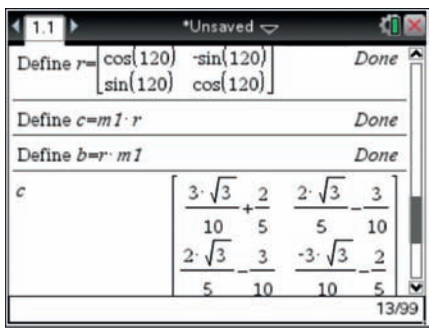


TI-Nspire CAS

Define m, p and q and then the reflection matrix. In the screen shot it is called $m1$ because the variable m has already been used for the slope.

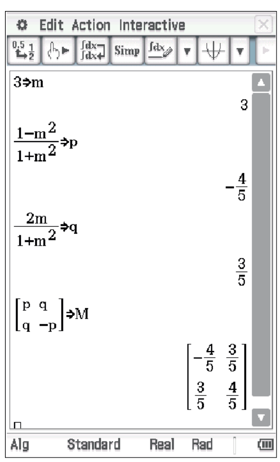


Make sure that your calculator is set on degrees.
Multiply the reflection by the rotation and vice versa, giving them different names. The screen shot shows the result for $\mathbf{M} \circ \mathbf{R}_{120°}$. You can check $\mathbf{R}_{120°} \circ \mathbf{M}$ and apply it to ABC on your own calculator.

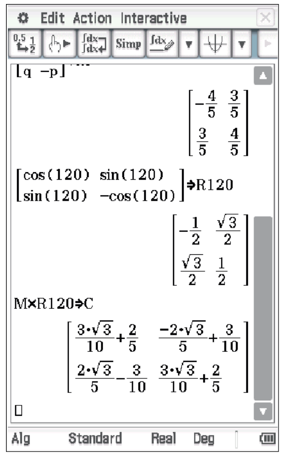


ClassPad

Ensure the calculator is set to **Standard**. Define m, p and q, and then the reflection matrix **M**. The Casio treats M and m as different variables.



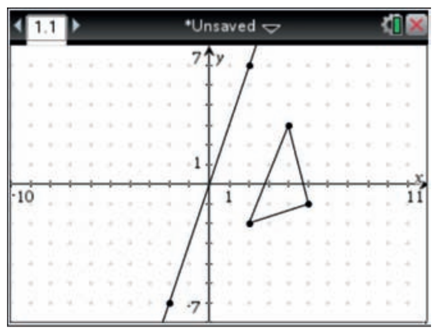
Make sure that your calculator is set on degrees.
Define the rotation matrix as **R**120.
Set **M** ×**R**120 to equal **C** and
set **R**120× **M** to equal **B**.
The screen shot shows the result for $\mathbf{M} \circ \mathbf{R}_{120°}$. You can check $\mathbf{R}_{120°} \circ \mathbf{M}$ and apply it to *ABC* on your own calculator.



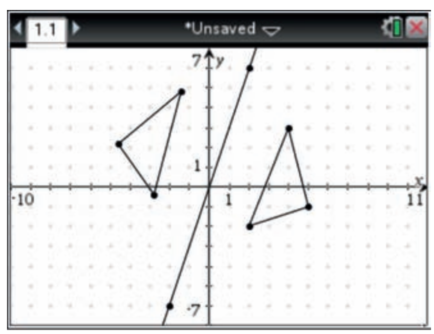
You could work out parts **d** and **c** of Example **15** using the geometry option on your CAS calculator as shown below. Unfortunately, current calculators will not do dilations unless both factors are the same, so you could not work out Example **14** in the same way. However, you could perform it with a general transformation on the CASIO ClassPad.

TI-Nspire CAS

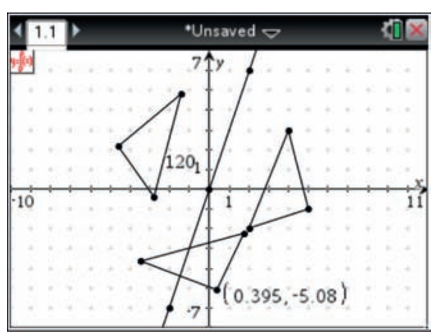
Draw the triangle on a Graph page (see page 417) with Window Settings of $-10 \le x \le 11$ and $-7 \le y \le 7$ for correct proportion. Draw a line through the origin with slope 3.



Use [menu], 8: Geometry, 5: Transformation and 2: Reflection to reflect *ABC* across the line by clicking on the triangle and clicking on the line. Use [menu], 9: Settings to ensure that the Graphing Angle and Geometry Angle are in degrees.

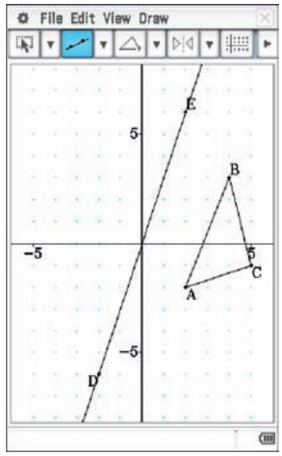


Now use [menu], 8: Geometry, 5: Transformation and 4: Rotation to rotate *ABC* around the origin by clicking on the origin and the reflected triangle and typing in the angle. You can find the coordinates of points using [menu], 1: Actions and 8: Coordinates and Equations.

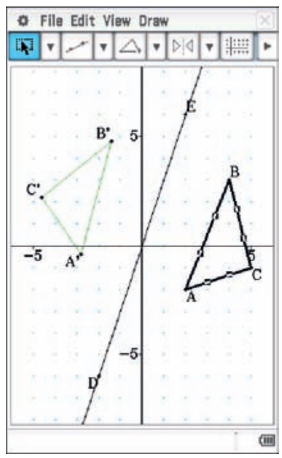


ClassPad

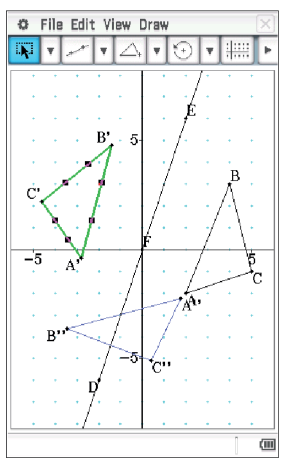
Use the **Geometry** to draw the triangle and the line of reflection.
Use **View Window** to set $-6 \le x \le 6$ and ymid = 0.
Refer to page 418.



Tap and select each side of the triangle.
Tap **Draw**, **Construct** and **Reflection**.
Tap the line of reflection.
The image, triangle $A'B'C'$, is drawn.



Make sure that your calculator is set to degrees by tapping and selecting **Geometry Format**.
Unselect the first triangle and select each side of the second (image) triangle.
Tap **Draw**, **Construct** and **Rotation**.
Tap the origin as the point of rotation.
Enter 120 as the angle of rotation.
The image, triangle $A''B''C''$, is drawn. It represents a reflection followed by a rotation of 120°.



EXERCISE 11.06 Composition of linear transformations

Concepts and techniques

1 Example 13 $\mathbf{T}_1$ is a translation of 3 to the right and 4 up. $\mathbf{T}_2$ is 2 to the left and 7 down and $\mathbf{T}_3$ is 4 to the right and 3 down.
   a Find $\mathbf{T}_1 \circ \mathbf{T}_2$ in the form $(x, y) \to (x', y')$.
   b Find $\mathbf{T}_3 \circ \mathbf{T}_2$ as a column vector.
   c Find the image of $A(-2, -1)$ $B(-1, 5)$ $C(6, 7)$ under the transformation $\mathbf{T}_1 \circ \mathbf{T}_3$.
   d Show ABC and $A'B'C'$ on the same set of axes.

2 $\mathbf{T}_1$ is a translation of 4 to the left and 2 up. $\mathbf{T}_2$ is 5 to the right and 4 down and $\mathbf{T}_3$ is 2 to the right and 4 up.
   a Find $\mathbf{T}_1 \circ \mathbf{T}_2$ in the form $(x, y) \to (x', y')$.
   b Find $\mathbf{T}_3 \circ \mathbf{T}_2$ as a column vector.
   c Find the image of $A(1, -4)$ $B(3, 2)$ $C(7, 5)$ $D(8, -2)$ under the transformation $\mathbf{T}_1 \circ \mathbf{T}_3$.
   d Show ABC and $A'B'C'$ on the same set of axes.

3 Example 14 a Find the composition $\mathbf{R}_{35°} \circ \mathbf{R}_{49°}$ of rotations of 35° and 49°, correct to 4 decimal places.
   b Find the composition $\mathbf{R}_{49°} \circ \mathbf{R}_{35°}$, correct to 4 decimal places.
   c Is this composition commutative?
   d Find the image of $A(3, 4)$ $B(5, 3)$ $C(2, 1)$ under $\mathbf{R}_{49°} \circ \mathbf{R}_{35°}$, correct to 2 decimal places.
   e Draw the object and image from part **d** on the same set of axes.

4 $\mathbf{M}_1$ is a reflection in $y = 3x$ and $\mathbf{M}_2$ is a reflection in $y = -2x$.
   a Find the composition $\mathbf{M}_1 \circ \mathbf{M}_2$.
   b Find the composition $\mathbf{M}_2 \circ \mathbf{M}_1$.
   c Is this composition commutative?
   d Find the image of $A(1, 5)$ $B(2, 3)$ $C(-2, 1)$ $D(-3, 4)$ under $\mathbf{M}_1 \circ \mathbf{M}_2$.
   e Draw the object and image from part **d** on the same set of axes.

5 Example 15 CAS $\mathbf{D}$ is a dilation by a factor of 1.5 in the x direction and 2 in the y direction. $\mathbf{M}$ is a reflection in a line of slope −2.
   a Find the composition $\mathbf{D} \circ \mathbf{M}$.
   b Find the composition $\mathbf{M} \circ \mathbf{D}$.
   c Is this composition commutative?
   d Find the image of $A(-2, 4)$ $B(1, 5)$ $C(3, 2)$ $D(-1, 0)$ under $\mathbf{M} \circ \mathbf{D}$.
   e Draw the object and image from part **d** on the same set of axes.

6 CAS $\mathbf{R}_{80°}$ is a rotation through 80° and $\mathbf{M}$ is a reflection in a 29° line of inclination.
   a Find the composition $\mathbf{R}_{80°} \circ \mathbf{M}$, correct to 4 decimal places.
   b Find the composition $\mathbf{M} \circ \mathbf{R}_{80°}$.
   c Is this composition commutative?
   d Find the image of $A(3, -1)$ $B(5, 1)$ $C(6, -2)$ $D(4, -3)$ under $\mathbf{R}_{80°} \circ \mathbf{M}$.
   e Draw the object and image on the same set of axes.

7 CAS $\mathbf{M}_1$ is a reflection in $y = x$ and $\mathbf{M}_2$ is a reflection in $y = -x$.

a Find the composition $\mathbf{M}_1 \circ \mathbf{M}_2$.

b Find the composition $\mathbf{M}_2 \circ \mathbf{M}_1$.

c Is this composition commutative?

Reasoning and communication

8 Show that a composition of rotations is always commutative by finding the compositions $\mathbf{R}_\alpha \circ \mathbf{R}_\beta$ and $\mathbf{R}_\beta \circ \mathbf{R}_\alpha$.

9 Show that composition of linear transformations is not always commutative by finding a counter example.

10 A general linear transformation is of the form $\mathbf{T}: (x, y) \rightarrow (ax + by, cx + dy)$, where a, b, c and d are real numbers. Write another general linear transformation $\mathbf{S}$ and show that the transformation $\mathbf{S} \circ \mathbf{T}$ is also linear.

11.07 INVERSE TRANSFORMATIONS

In the previous sections we covered various transformations on the plane (2D space). Some of these transformations can be reversed by another transformation so that the combined effect is to make no change, i.e., $(x, y) \rightarrow (x, y)$. This is like adding 0 or multiplying by 1.

IMPORTANT

If it exists, the element that leaves everything unchanged for an operation is called the **identity** for that operation and is usually written as 1, 0 or **I**.

For all transformations, the identity is $\mathbf{I}: (x, y) \rightarrow (x, y)$.

For translations, the identity is none along and none up or down, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The identity matrix representation for linear transformations is $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

For rotations, it is a rotation though 0°, $\mathbf{R}_0$, (or 360°) or the equivalent in radians.

For dilations, the identity has a factor of 1 in both the x and y directions.

There is no reflection that is equal to the identity.

A transformation that reverses a particular transformation so that their composition is the identity has a special name.

IMPORTANT

For a transformation $\mathbf{T}$, if there exists a transformation $\mathbf{S}$ such that

$\mathbf{S} \circ \mathbf{T} = \mathbf{T} \circ \mathbf{S} = \mathbf{I}$

then $\mathbf{S}$ is called the **inverse** of $\mathbf{T}$ and written as $\mathbf{T}^{-1}$ so $\mathbf{T}^{-1} \circ \mathbf{T} = \mathbf{T} \circ \mathbf{T}^{-1} = \mathbf{I}$.

A transformation that has an inverse is called an **invertible** transformation.

You have already seen that some compositions of translations and products of matrices are not commutative, which is the reason why the identity and inverse have to work on *both the left and right*. It is obvious that the inverse of a translation is another translation in the opposite direction. The inverse of a dilation is another dilation with factors that are reciprocals of the factors of the original dilation. The inverse of a rotation is an equal rotation in the opposite direction.

Example 16

Find the inverses of the following transformations and confirm that they are correct by a composition of the transformation and its inverse.

- **a** A translation of 5 up and 3 left.
- **b** A dilation by a factor of 0.5 in the x direction and 0.8 in the y direction.
- **c** A rotation through an angle of $\frac{3\pi}{4}$.

Solution

a Write the translation.

$\mathbf{T}: (x, y) \rightarrow (x - 3, y + 5)$

Write the inverse, $\mathbf{T}^{-1}$.

$\mathbf{T}^{-1}: (x, y) \rightarrow (x + 3, y - 5)$

Use vectors to do the composition on the left.

$$\begin{aligned}\mathbf{T}^{-1} \circ \mathbf{T} &= \mathbf{T}^{-1} + \mathbf{T}\\ &= \begin{bmatrix}-5\\3\end{bmatrix} + \begin{bmatrix}5\\-3\end{bmatrix}\\ &= \begin{bmatrix}-5+5\\3+(-3)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \checkmark\end{aligned}$$

Do the composition on the right.

$$\begin{aligned}\mathbf{T} \circ \mathbf{T}^{-1} &= \mathbf{T} + \mathbf{T}^{-1}\\ &= \begin{bmatrix}5\\-3\end{bmatrix} + \begin{bmatrix}-5\\3\end{bmatrix}\\ &= \begin{bmatrix}5+(-5)\\-3+3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \checkmark\end{aligned}$$

b Write the dilation.

$\mathbf{D}: (x, y) \rightarrow (0.5x, 0.8y)$

Use $\frac{1}{0.5} = 2$ and $\frac{1}{0.8} = 1.25$ for $\mathbf{D}^{-1}$.

$\mathbf{D}^{-1}: (x, y) \rightarrow (2x, 1.25y)$

Use the matrix representations to check the composition on the left.

$$\begin{aligned}\mathbf{D}^{-1} \circ \mathbf{D} &= \begin{bmatrix}2 & 0\\0 & 1.25\end{bmatrix}\begin{bmatrix}0.5 & 0\\0 & 0.8\end{bmatrix}\\ &= \begin{bmatrix}2\times 0.5 & 0\\0 & 0.8\times 1.25\end{bmatrix}\\ &= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \checkmark\end{aligned}$$

Now repeat it on the right.

$$\begin{aligned}\mathbf{D} \circ \mathbf{D}^{-1} &= \begin{bmatrix}0.5 & 0\\0 & 0.8\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 1.25\end{bmatrix}\\ &= \begin{bmatrix}0.5\times 2 & 0\\0 & 0.8\times 1.25\end{bmatrix}\\ &= \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \checkmark\end{aligned}$$

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c The inverse will be a rotation of the same amount in the opposite direction.

Angle of rotation for the inverse $= -\frac{3\pi}{4}$

Write the matrix for the rotation.

$$\mathbf{R} = \begin{bmatrix} \cos\left(\frac{3\pi}{4}\right) & -\sin\left(\frac{3\pi}{4}\right) \\ \sin\left(\frac{3\pi}{4}\right) & \cos\left(\frac{3\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Write the matrix for the inverse.

$$\mathbf{R}^{-1} = \begin{bmatrix} \cos\left(-\frac{3\pi}{4}\right) & -\sin\left(-\frac{3\pi}{4}\right) \\ \sin\left(-\frac{3\pi}{4}\right) & \cos\left(-\frac{3\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Use matrix multiplication to check the composition on the left.

$$\mathbf{R}^{-1} \circ \mathbf{R} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & -\frac{1}{2}+\frac{1}{2} \\ -\frac{1}{2}+\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Now repeat it on the right.

$$\mathbf{R} \circ \mathbf{R}^{-1} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

The inverses of translations and rotations are opposite to each other. However, the inverses of reflections are different. How do you reverse a reflection? With a mirror, you use another mirror. The inverse of a reflection is the same reflection. *Reflections are their own inverses.*

Example 17

Use matrices to confirm that the following reflections are their own inverses.

a The reflection in the line with slope –0.6.

b The reflection in the line with an inclination angle of 30° to the x-axis.

Solution

a Find p.

$$p = \frac{1-m^2}{1+m^2} = \frac{1-(-0.6)^2}{1+(-0.6)^2} = \frac{8}{17} \approx 0.4706$$

Find q. $q = \frac{2m}{1+m^2} = \frac{2\times -0.6}{1+0.36} = -\frac{15}{17} \approx -0.8824$

Write the matrix for $\mathbf{M}$. $\mathbf{M} = \begin{bmatrix} p & q \\ q & -p \end{bmatrix} = \begin{bmatrix} \frac{8}{17} & -\frac{15}{17} \\ -\frac{15}{17} & -\frac{8}{17} \end{bmatrix}$

The inverse will be the same. $\mathbf{M}^{-1} = \begin{bmatrix} \frac{8}{17} & -\frac{15}{17} \\ -\frac{15}{17} & -\frac{8}{17} \end{bmatrix}$

Use matrix multiplication to check the composition on the left.
Since the matrices are the same it is not necessary to repeat it on the right as well.

$$\mathbf{M} \times \mathbf{M}^{-1} = \begin{bmatrix} \frac{8}{17} & -\frac{15}{17} \\ -\frac{15}{17} & -\frac{8}{17} \end{bmatrix}\begin{bmatrix} \frac{8}{17} & -\frac{15}{17} \\ -\frac{15}{17} & -\frac{8}{17} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{64+225}{289} & \frac{-120+120}{289} \\ \frac{120-120}{289} & \frac{64+225}{289} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

b Write the matrix for $\mathbf{M}$. $\mathbf{M} = \begin{bmatrix} \cos(60°) & \sin(60°) \\ \sin(60°) & -\cos(60°) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

The inverse will be the same. $\mathbf{M}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

Use matrix multiplication to check the composition on the left.
Since the matrices are the same it is not necessary to repeat it on the right as well.

$$\mathbf{M} \times \mathbf{M}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{4}+\frac{3}{4} & \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} & \frac{1}{4}+\frac{3}{4} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

You can also use your CAS calculator to find the inverse of a square matrix **A** using **A-1** in the same way as for ordinary numbers.

What is the inverse of a composition of two transformations? Is it the composition of the inverses? Does it depend on whether or not the transformations commute?

INVESTIGATION Inverses of compositions

Consider the following pairs of matrices:

$\mathbf{T}_1$ is a rotation of $\frac{\pi}{3}$ and $\mathbf{T}_2$ is a reflection in the line with inclination $-\frac{\pi}{8}$.

$\mathbf{T}_1$ is a reflection in the line with slope 15° and $\mathbf{T}_2$ is a dilation with factor $\frac{2}{3}$ in both directions.

Use matrices or other means to investigate the compositions $\mathbf{T}_1 \circ \mathbf{T}_2$, $\mathbf{T}_2 \circ \mathbf{T}_1$, $(\mathbf{T}_1 \circ \mathbf{T}_2)^{-1}$, $(\mathbf{T}_2 \circ \mathbf{T}_1)^{-1}$, $\mathbf{T}_1^{-1} \circ \mathbf{T}_2^{-1}$ and $\mathbf{T}_2^{-1} \circ \mathbf{T}_1^{-1}$.

What do you find?

Try inverses of compositions of other linear transformations.

Can you make a general rule?

Can you prove it?

From your investigation, you would have found the following.

IMPORTANT

If it exists, the inverse of a composition of linear transformations, $\mathbf{T}_1 \circ \mathbf{T}_2$, is given by $\mathbf{T}_2^{-1} \circ \mathbf{T}_1^{-1}$.

EXERCISE 11.07 Inverse transformations

Concepts and techniques

1 Example 16 For each of the following translations, write the translation as a vector, find its inverse and verify your results by composing them.

a 5 up and 3 right
b 4 down and 5 left
c 2 right and 6 down
d 7 left and 5 up

2 Example 17 Find the matrix for dilation with a factor of $\frac{1}{3}$ in the x direction and $\frac{1}{2}$ in the y direction and its inverse and verify your results by composing them.

3 Find the matrix for dilation with a factor of $2\frac{1}{2}$ and its inverse and verify your results by composition.

4 Find the matrix for rotation through an angle of $\frac{4\pi}{3}$ and its inverse and verify your results by composition.

5 Find the matrix for rotation through an angle of 120° and its inverse and verify your results by composition.

6 Find the matrix for reflection through a line with an inclination of 112.5° and its inverse and verify your results by composition.

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7 Find the matrix for reflection through a line with slope $-\frac{1}{3}$ and its inverse and verify your results by composition.

8 For each of the following pairs of transformations, find $\mathbf{T}_1 \circ \mathbf{T}_2$, $(\mathbf{T}_1 \circ \mathbf{T}_2)^{-1}$, $\mathbf{T}_2^{-1} \circ \mathbf{T}_1^{-1}$ and $\mathbf{T}_1^{-1} \circ \mathbf{T}_2^{-1}$ using matrices, and comment on the results.

a A rotation through an angle of 30° and a rotation through an angle of 45°.

b A reflection through a line with slope 3 and a reflection through a line with slope $\frac{1}{2}$.

Reasoning and communication

9 $\mathbf{T}_1$ and $\mathbf{T}_2$ are the matrices of invertible linear transformations and $\mathbf{T}_3 = \mathbf{T}_1 \circ \mathbf{T}_2$.

a What are the products of $\mathbf{T}_1$ and its inverse and vice versa?

b What are the products of $\mathbf{T}_2$ and its inverse and vice versa?

c What is the matrix for $\mathbf{T}_1^{-1} \circ \mathbf{T}_3$?

d By what does the matrix for $\mathbf{T}_1^{-1} \circ \mathbf{T}_3$ have to be multiplied on the left to give **I**?

e Use the results from parts **a** to **d** to prove that $(\mathbf{T}_1 \circ \mathbf{T}_2)^{-1} = \mathbf{T}_2^{-1} \circ \mathbf{T}_1^{-1}$ for all invertible linear transformations.

10 Prove that for linear transformations that commute, $(\mathbf{T}_1 \circ \mathbf{T}_2)^{-1} = \mathbf{T}_1^{-1} \circ \mathbf{T}_2^{-1}$.

11.08 DETERMINANTS AND GEOMETRY

In Chapter 8, you saw that invertible matrices had a non zero **determinant**. The formula for the determinant of a square matrix of order 2 is shown on the right.

Any linear transformation can be written as a 2×2 matrix, so you can calculate the determinant of its matrix.

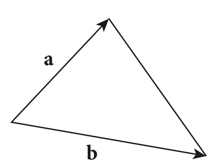
Finding the area of a closed plane figure is sometimes important. The simplest way to find the area for many polygons is to divide them into triangles. There are many ways to work out the area of a triangle, including the use of vector properties as shown on the opposite page.

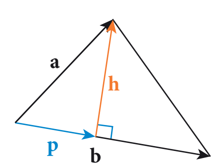
IMPORTANT

The determinant of the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by

$\det \mathbf{A} = |\mathbf{A}| = ad - bc$

Consider the triangle formed by two vectors **a** and **b** as shown below in the diagram on the left.





Construct the orthogonal vectors **p** and **h** as shown on the right. The area of the triangle is then given by $\frac{1}{2}bh$. **p** is the projection of **a** on **b** and $\mathbf{a} = \mathbf{p} + \mathbf{h}$ so $\mathbf{h} = \mathbf{a} - \mathbf{p}$. As you saw in Chapter 4 (see page 128), the projection of **a** on **b** is given by $\mathbf{p} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\hat{\mathbf{b}} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} \times \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$.

Thus $\mathbf{h} = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$ and you can use $|\mathbf{b}|$ and $|\mathbf{h}|$ to find the area of the triangle.

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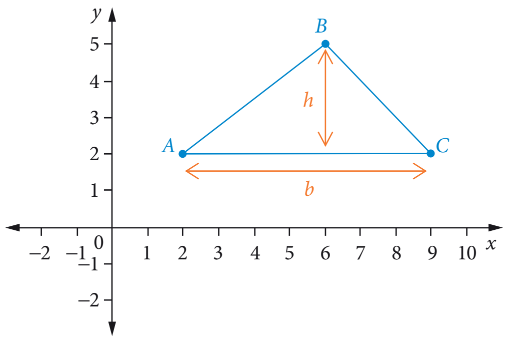
Example 18

a Find the area of the triangle $A(2, 2)$ $B(6, 5)$ $C(9, 2)$.
b Find the matrix for the transformation $\mathbf{T}$: $(x, y) \to (2x + 3y, -2x + 2y)$.
c Find the image $A'B'C'$ of the triangle under the transformation $\mathbf{T}$.
d Find the area of $A'B'C'$.
e Find the determinant of the matrix of $\mathbf{T}$.
f Comment on your results.

Solution

a Sketch ABC.
To find the area, you can use $A = \frac{1}{2}bh$



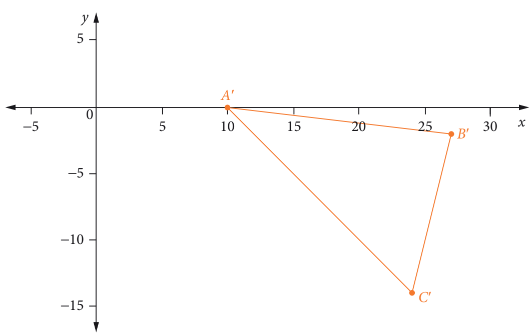
Work out the area. Area of $ABC = \frac{1}{2}bh = \frac{1}{2} \times 7 \times 3 = 10.5$

b Write the matrix for the transformation. $\mathbf{T} = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix}$

c Use the matrix to find $A'B'C'$. $\begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 6 & 9 \\ 2 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 27 & 24 \\ 0 & -2 & -14 \end{bmatrix}$

Write the answer. The image is $A'(10, 0)$ $B'(27, -2)$ $C'(24, -14)$.

d Sketch $A'B'C'$.
No sides are vertical or horizontal, so use vectors or the semiperimeter formula to find the area.

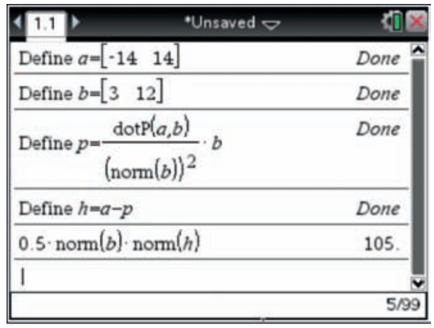


Use the vectors $\mathbf{C'A'}$ and $\mathbf{C'B'}$ to find the area.

$\mathbf{C'A'} = (10 - 24, 0 - (-14)) = (-14, 14)$
$\mathbf{C'B'} = (27 - 24, -2 - (-14)) = (3, 12)$

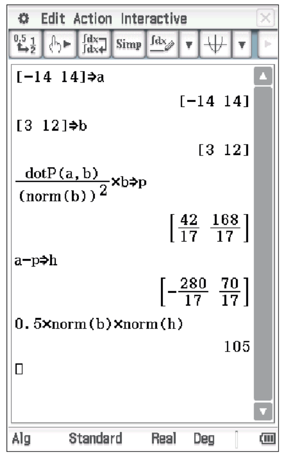
TI-Nspire CAS

Use your calculator to find the projection **p** and hence the height **h** and area of the triangle.



ClassPad

Set the calculator to **Standard** mode. Use your calculator to find the projection **p** and hence the height **h** and area of the triangle.



Write the area. Area of $A'B'C' = 105$

d Find the determinant. $\det \mathbf{T} = ad - bc = 2 \times 2 - 3 \times (-2) = 10$

e Compare the areas. $\text{Area}_{ABC} = 10.5$ and $\text{Area}_{A'B'C'} = 105$

f Write a comment. The area of the image is $\det \mathbf{T} = 10$ times the area of the object.

From Example **18**, it appears that the area of the object under the linear transformation **T** is multiplied by $|\mathbf{T}|$ to get the area of the image. This can be proved for the general case.

IMPORTANT

The area of the image produced by a linear transformation is $|\mathbf{T}|$ times the area of the object.

Of course, if **T** is negative then you need to change the sign of $|\mathbf{T}|A$ because an area is always positive. What is the significance of the negative sign of the determinant for the image compared to the object?

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Example 19

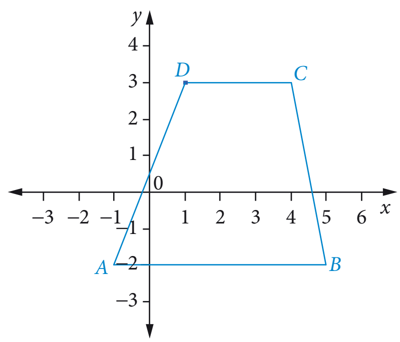
A linear transformation is given by $\mathbf{T}$: $(x, y) \rightarrow (3x - y, 2x + 3y)$. What is the area of the new shape of the trapezium $A(-1, -2)$ $B(5, -2)$ $C(4, 3)$ $D(1, 3)$ under the transformation?

Solution

Write the matrix for the transformation. $\mathbf{T} = \begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix}$

Find the determinant of $\mathbf{T}$. $|\mathbf{T}| = ad - bc = 3 \times 3 - (-1) \times 2 = 11$

Sketch the trapezium.



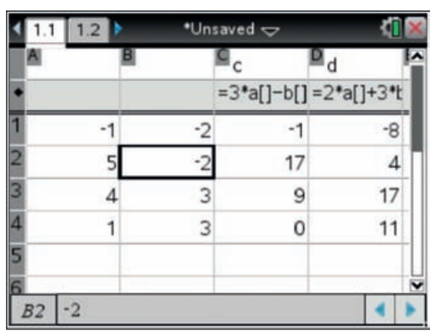
Find the area of the trapezium. $A = \frac{1}{2}(a+b)h = \frac{1}{2}(6+3)\times 5 = 22.5$

Multiply by the determinant. Area of new shape $= 22.5 \times 11 = 247.5$

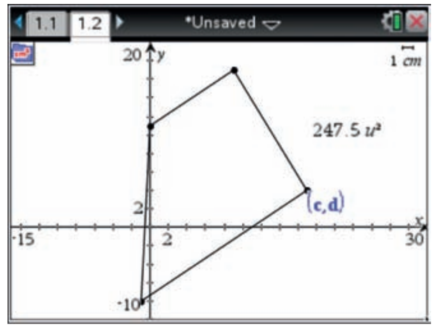
You can use your CAS calculator to perform this transformation and find the area of the transformed shape. You can also use the det() function on the TI-Nspire CAS or the ClassPad to find the determinant of a matrix.

TI-Nspire CAS

Put the points into a Spreadsheet page and apply the transformation (see pages 422–423). Let the variables for columns C and D be c and d, say.



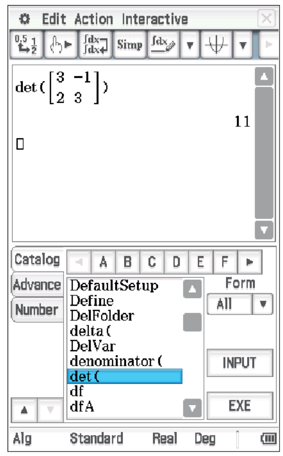
Add a Graph page and make the Window Settings $-15 \le x \le 30$ and $-10 \le y \le 20$ to fit the new shape (see pages 411-412). Now use menu, 3: Graph Entry/Edit, 5: Scatter Plot and put $x \leftarrow c$ and $y \leftarrow d$ in the dialog box. Draw a polygon joining up the points. You can find the area using menu, 8: Geometry, 3: Measurement, 2: Area and clicking on the polygon.



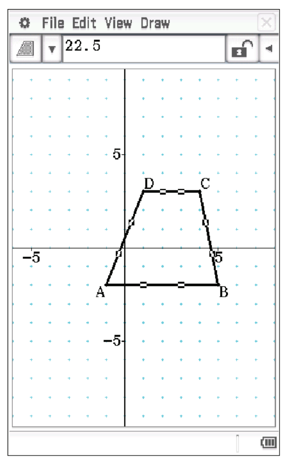
The display digits in 9: Settings have been changed to Float 4.

ClassPad

Use the Main $\sqrt{\alpha}$ application to calculate the determinant.



You can do the rest from the **Geometry** application.
First draw the trapezium.
Start with **View Window** set to about $-6 \le x \le 8$ and ymid = 0.
Select tap each side and tap to measure its area. Make sure that the area symbol is displayed in the top left drop down menu.



Area = 22.5

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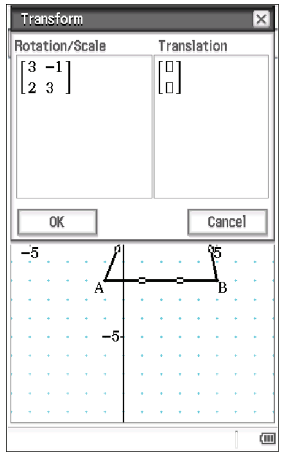
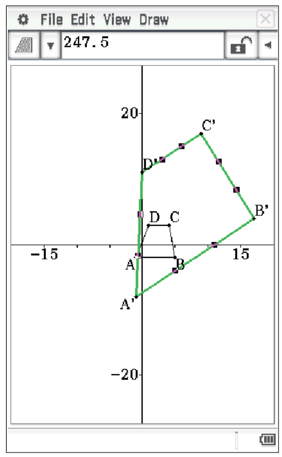
Tap **Draw**, then **Construct**, then **General Transform**. Fill in the 2 × 2 matrix only.
Tap **OK**.



Change **View Window** so $-20 \le x \le 20$.
The dots disappear.
First tap a blank area of screen if it is not clear whether the small original trapezium is still selected. Select each side of the (larger) image and find its area.



What kind of transformation do you think the combination of two dilations gives? What about two rotations? What about two reflections?

Example 20

a Find the result of applying the reflection in a line of inclination 30°, followed by a reflection in a line of slope 45°.
b What kind of transformation is the composition of the two reflections?

Solution

a Write the matrix for the first reflection.

$$\mathbf{M}_1 = \begin{bmatrix} \cos(60°) & \sin(60°) \\ \sin(60°) & -\cos(60°) \end{bmatrix}$$

Write the matrix for the second reflection.

$$\mathbf{M}_2 = \begin{bmatrix} \cos(90°) & \sin(90°) \\ \sin(90°) & -\cos(90°) \end{bmatrix}$$

Find the matrix for $\mathbf{M}_2 \circ \mathbf{M}_1$.

$$\mathbf{M}_2 \times \mathbf{M}_1 = \begin{bmatrix} \cos(90°) & \sin(90°) \\ \sin(90°) & -\cos(90°) \end{bmatrix}\begin{bmatrix} \cos(60°) & \sin(60°) \\ \sin(60°) & -\cos(60°) \end{bmatrix}$$

Multiply.

$$= \begin{bmatrix} \cos(90°)\cos(60°) & \cos(90°)\sin(60°) \\ +\sin(90°)\sin(60°) & -\sin(90°)\cos(60°) \\ & \\ \sin(90°)\cos(60°) & \cos(90°)\cos(60°) \\ -\cos(90°)\sin(60°) & +\sin(90°)\sin(60°) \end{bmatrix}$$

b Rearrange the top right term of the matrix.

$$= \begin{bmatrix} \cos(90°)\cos(60°) & -[\sin(90°)\cos(60°) \\ +\sin(90°)\sin(60°) & -\cos(90°)\sin(60°)] \\ & \\ \sin(90°)\cos(60°) & \cos(90°)\cos(60°) \\ -\cos(90°)\sin(60°) & +\sin(90°)\sin(60°) \end{bmatrix}$$

Use the trigonometric addition formulas in reverse.

$$= \begin{bmatrix} \cos(90°-60°) & -\sin(90°-60°) \\ \sin(90°-60°) & \cos(90°-60°) \end{bmatrix}$$

Simplify.

$$= \begin{bmatrix} \cos(30°) & -\sin(30°) \\ \sin(30°) & \cos(30°) \end{bmatrix}$$

But this is a rotation. $= \mathbf{R}_{30°}$

Write the answer. A reflection in a line of inclination 30°, followed by a reflection in a line of slope 45° gives a rotation of 30°.

You can use matrices to prove that the result of Example **21** is generally true.

IMPORTANT

The composition of two reflections in lines through the origin gives a rotation around the origin.

EXERCISE 11.08 Determinants and geometry

Concepts and techniques

WS

Matrix multiplications

Your teacher will tell you where you should use your calculator in this exercise.

1 Example 19 Find the area of each shape below under the given transformation.

a The rectangle $A(3, 2)$ $B(5, 2)$ $C(5, -5)$ $D(3, -5)$ under the transformation **T**: $(x, y) \rightarrow (3x - 2x, 2x + 3y)$.

b The square $A(3, 8)$ $B(-3, 8)$ $C(-3, 2)$ $D(3, 2)$ under the transformation **T**: $(x, y) \rightarrow (x - 3y, x + 7y)$.

c The kite $A(2, 5)$ $B(-2, 1)$ $C(2, -10)$ $D(6, 1)$ under the transformation **T**: $(x, y) \rightarrow (2x + y, x + 3y)$.

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2 Example 20 What is result of a reflection in the x-axis followed by a rotation of 90°?

3 What is the rotation of –90° followed by a rotation of 45°?

4 What is the result of a dilation of factors 2 and 3, followed by a dilation of factors 4 and 2, where the factors are given in the order x then y?

5 What is the result of a rotation of 90° followed by a reflection in the y-axis?

Reasoning and communication

6 Example 18 a Find the area of the triangle $A(-1, -2)$ $B(2, 2)$ $C(4, 2)$.
 b Find the matrix for the transformation $\mathbf{T}$: $(x, y) \to (x - 2y, 2x + 3y)$.
 c Find the image $A'B'C'$ of the triangle under the transformation $\mathbf{T}$.
 d Find the area of $A'B'C'$.
 e Find the determinant of the matrix of $\mathbf{T}$.
 f Comment on your results.

7 a Find the area of the rectangle $A(3, -3)$ $B(5, 0)$ $C(-1, 4)$ $D(-3, 1)$.
 b Find the matrix for the transformation $\mathbf{T}$: $(x, y) \to (y - 2x, 2x - 2y)$.
 c What will be the area of the new shape under the transformation $\mathbf{T}$?

8 Find the area of each shape below under the given transformation.
 a The triangle $A(-3, -2)$ $B(-3, 6)$ $C(4, 1)$ under the transformation $\mathbf{T}$: $(x, y) \to (2y - 3x, x + y)$.
 b The rectangle $A(1, -5)$ $B(5, -2)$ $C(-1, 6)$ $D(-5, 3)$ under the transformation $\mathbf{T}$: $(x, y) \to (2x + 4y, 3x + 5y)$.
 c The kite $A(3, 10)$ $B(9, 8)$ $C(7, 2)$ $D(-3, 2)$ under the transformation $\mathbf{T}$: $(x, y) \to (3x + 5y, 3x - y)$.

9 a Find the result of applying a reflection in the line through the origin of inclination 40°, followed by a reflection in the line through the origin of slope 15°.
 b What kind of transformation is the composition of the two reflections?

10 a Find the result of applying the reflection in a line of slope –2, followed by a reflection in a line of slope 3.
 b What kind of transformation is the composition of the two reflections?

11 Prove that the composition of two dilations is another dilation.

12 Prove that the composition of two reflections across a line through the origin is a rotation.

11

CHAPTER SUMMARY

TRANSFORMATIONS IN THE PLANE

- A **translation** is a *slide* of all the points in the plane the same distance in the same direction.
- A translation can be modelled as addition of column matrices. The column matrices representing points are translated to the new points by adding the column matrix representing the translation.
- A **linear transformation of the plane** changes the point $P(x, y)$ to $P'(ax + by, cx + dy)$, where a, b, c and d are constants (real numbers).
- The linear transformation that changes the point $P(x, y) \to P'(ax + by, cx + dy)$ is modelled by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.
- Linear transformations are usually written as $\mathbf{T} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The image of a vector $\mathbf{v}$ under a transformation $\mathbf{T}$ is written as $\mathbf{Tv}$.
- A simple **dilation** *stretches* or *compresses* the points in the plane in the x and y directions. The dilation by a factor of λ_1 in the x direction and λ_2 in the y direction is given as $(x, y) \to (\lambda_1 x, \lambda_2 y)$.
- Dilation by factors of λ_1 and λ_2 in the x and y directions respectively is modelled by the matrix $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.
- For $\lambda_1, \lambda_2 > 1$, the dilation is a magnification that makes shapes larger. For $\lambda_1, \lambda_2 < 1$ the dilation is a reduction that makes shapes smaller.
- A simple **rotation** *turns* all points through the same angle around the origin. The ***rotation*** through an angle α around the origin is given by $(x, y) \to (x \cos(\alpha) - y \sin(\alpha), x \sin(\alpha) + y \cos(\alpha))$, where α is the usual anticlockwise rotation. For a clockwise rotation, α is negative.
- **Rotation** through an angle α around the origin is modelled by the matrix $\mathbf{R}_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$.
- A simple **reflection** swaps all points across to the opposite side of a line through the origin. The line may be expressed by its inclination angle, gradient or equation. The reflection across a line with slope m through the origin is given by $(x, y) \to$ $\left(\left(\frac{1-m^2}{1+m^2}\right)x + \left(\frac{2m}{1+m^2}\right)y, \left(\frac{2m}{1+m^2}\right)x - \left(\frac{1-m^2}{1+m^2}\right)y\right)$ or $(x, y) \to (px + qy, qx - py)$, where $p = \frac{1-m^2}{1+m^2}$ and $q = \frac{2m}{1+m^2}$. The reflection across a line of inclination θ through the origin is given by $(x, y) \to (\cos(2\theta)x + \sin(2\theta)y, \sin(2\theta)x - \cos(2\theta)y)$
- **Reflection** through a line at a slope of m or inclination $\theta = \tan^{-1}(m)$ through the origin is modelled by the matrix $\mathbf{M} = \begin{bmatrix} p & q \\ q & -p \end{bmatrix}$ where $p = \frac{1-m^2}{1+m^2}$ and $q = \frac{2m}{1+m^2}$ or $\mathbf{M} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

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- The **composition** of the transformations **S** and **T** is the combined transformation such that if **T**: $(x, y) \to (x', y')$ and **S**: $(x', y') \to (x'', y'')$ then the composition is $\mathbf{S} \circ \mathbf{T}$: $(x, y) \to (x', y') \to (x'', y'')$.
- The composition $\mathbf{S} \circ \mathbf{T}$ is given by the matrix product **ST**. In general, composition of transformations is not commutative.
- If it exists, the element that leaves everything unchanged for an operation is called the **identity** for that operation and is usually written as 1, 0 or **I**. For all transformations, the identity is **I**: $(x, y) \to (x, y)$.
- For translations the identity is none along and none up or down, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
- For rotations the identity is a rotation though 0°, $\mathbf{R}_0$, (or 360°) or the equivalents in radians.
- For dilations the identity has both factors equal to 1.
- There is no reflection equal to the identity.
- The matrix representation for **I** for dilations, rotations and reflections is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- For a transformation **T**, if there exists a transformation **S** such that $\mathbf{S} \circ \mathbf{T} = \mathbf{T} \circ \mathbf{S} = \mathbf{I}$

  then **S** is called the **inverse** of **T** and written as $\mathbf{T}^{-1}$ so $\mathbf{T}^{-1} \circ \mathbf{T} = \mathbf{T} \circ \mathbf{T}^{-1} = \mathbf{I}$.
- A translation that has an inverse is called an **invertible** transformation.
- The inverse of a translation $\begin{bmatrix} a \\ b \end{bmatrix}$ is the translation $\begin{bmatrix} -a \\ -b \end{bmatrix}$ in the opposite direction.
- The inverse of a dilation $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ is the dilation $\mathbf{D}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{bmatrix}$ with reciprocal factors in the x and y directions.
- The inverse of a rotation $\mathbf{R}_\alpha = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$ is the rotation $(\mathbf{R}_\alpha)^{-1} = \mathbf{R}_{-\alpha} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix}$ in the opposite direction.
- Reflections are their own inverses.
- The determinant of the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\det \mathbf{A} = |\mathbf{A}| = ad - bc$.
- The area of the image produced by a linear transformation **T** is $|\mathbf{T}|$ times the area of the object.
- If it exists, the inverse of a composition of linear transformations, $\mathbf{T}_1 \circ \mathbf{T}_2$, is given by $\mathbf{T}_2^{-1} \circ \mathbf{T}_1^{-1}$.

11 CHAPTER REVIEW
TRANSFORMATIONS IN THE PLANE

Multiple choice

1 Example 1 The image of the triangle $A(-2, -3)$ $B(1, 4)$ $C(2, -2)$ under the translation 5 up and 3 left is:

A $A'(3, -6)$ $B'(6, 1)$ $C'(7, 1)$
B $A'(3, 0)$ $B'(6, 7)$ $C'(7, 1)$
C $A'(-5, 2)$ $B'(-2, 9)$ $C'(-1, 3)$
D $A'(1, 2)$ $B'(4, 9)$ $C'(5, 3)$
E $A'(-7, -6)$ $B'(-4, 1)$ $C'(-3, -5)$

2 Example 4 What is the matrix for $\mathbf{T}$: $(x, y) \to (2y - 5x, 4x + y)$?

A $\begin{bmatrix} 2 & -5 \\ 4 & 1 \end{bmatrix}$
B $\begin{bmatrix} 2 & 4 \\ -5 & 1 \end{bmatrix}$
C $\begin{bmatrix} 2 & -5 \\ 4 & 1 \end{bmatrix}$
D $\begin{bmatrix} -5 & 4 \\ 2 & 1 \end{bmatrix}$
E $\begin{bmatrix} -5 & 2 \\ 4 & 1 \end{bmatrix}$

3 Example 7 What is the matrix for a dilation by a factor of 2 in the x direction and 3 in the y direction?

A $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$
B $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
C $\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$
D $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$
E $\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

4 Example 9 Example 10 What is the image of the point (2, 6) after a rotation of $\frac{2\pi}{3}$ around the origin?

A $(-3\sqrt{3} - 1, \sqrt{3} - 3)$
B $(1 - 3\sqrt{3}, 3 - \sqrt{3})$
C $(\sqrt{3} - 1, \sqrt{3} + 1)$
D $(1 - 3\sqrt{3}, 3\sqrt{3} + 1)$
E $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

5 Example 12 Example 17 What is the matrix for the inverse of the reflection through the line $y = 2x$?

A $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$
B $\begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$
C $\begin{bmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{bmatrix}$
D $\begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}$
E $\begin{bmatrix} 0.8 & -0.6 \\ -0.6 & -0.8 \end{bmatrix}$

6 Example 14 Example 15 Which of the following is true about compositions?

A Dilations and reflections commute
B No transformations commute
C All transformations commute
D None of the above
E Two rotations make a reflection

Short answer

7 Example 2 Use column matrices to find the points $A(-3, 5)$ $B(1, 7)$ $C(4, 3)$ $D(8, -2)$ after they are transformed by a translation of 3 down and 7 right.

8 Example 3 a Show that the quadrilateral $A(-4, 3)$ $B(3, 3)$ $C(2, -1)$ $D(-5, -1)$ is a parallelogram.
b Find the image of the parallelogram under the transformation $(x, y) \rightarrow (3x + 3y, 2x + 4y)$.
c What is the shape of $A'B'C'D'$?

9 Example 5 Example 6 Example 7 a What is the transformation for a dilation **D** of factor 3 in the x direction and 1.5 in the y direction?
b What is the matrix for the dilation **D**?
c Find the image of the right-angled triangle $A(-3, -2)$ $B(3, 6)$ $C(-1, 9)$ under the dilation **D**.
d What is the shape of the image?

10 Example 8 Example 9 a What is the matrix for a rotation of 150° around the origin?
b Find the image of the rectangle $A(-1, -2)$ $B(4, -2)$ $C(4, 5)$ $D(-1, 5)$ under the rotation.
c What is the relationship between the object and the image?

11 Example 12 Find the image (correct to two decimal places) of the quadrilateral $A(-3, 4)$ $B(-1, 8)$ $C(2, 7)$ $D(4, 3)$ after reflection across a line through the origin of inclination $\frac{7\pi}{8}$.

12 Example 13 Find the image of the triangle $A(-2, 4)$ $B(7, 2)$ $C(5, 1)$ after application of the transformation $\mathbf{S} \circ \mathbf{T}$, where **S** is the translation 2 right and 4 down and **T** is the translation 5 right and 6 up.

13 Example 14 Example 15 a Find the image of the triangle $A(-2, -4)$ $B(1, 5)$ $C(3, 2)$ after the transformation $\mathbf{D} \circ \mathbf{M}$, where **M** is the reflection in the line through the origin with slope $-\frac{1}{3}$ and **D** is the dilation with an x factor of 2 and a y factor of 1.5.
b Use a calculator or other means to determine whether **D** and **M** commute.

14 Example 16 Example 17 Find the inverses of the following transformations.
a A translation 3 left and 4 up.
b A dilation with an x factor of $\frac{1}{2}$ and a y factor of 1.5.
c A rotation of $\frac{4\pi}{3}$.
d A reflection in the line through the origin with an inclination of 25°.

15 Example 19 Find the area of the image of the rectangle $A(-3, -4)$ $B(-3, 5)$ $C(2, 5)$ $D(2, -4)$ after the transformation $(x, y) \rightarrow (4x - y, x + y)$.

Application

16 Use vectors or other means to show that translation is associative under composition. That is, for any translations **S**, **R** and **T**, $\mathbf{S} \circ (\mathbf{R} \circ \mathbf{T}) = (\mathbf{S} \circ \mathbf{R}) \circ \mathbf{T}$.

17 a Find the area of the triangle $A(2, 6)$ $B(5, 2)$ $C(3, 1)$.
b Show that the image in **a**, after transformation by a dilation with an x factor of 2 and a y factor of 3.5, has an area that is 7 times that of the original triangle.

Practice quiz

18 Use matrices or other methods to prove that the composition of two rotations is another rotation.